Commun. Math. Phys. 286, 399–443 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-008-0705-0

Communications in
Mathematical Physics

BFV-Complex and Higher Homotopy Structures

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Received: 10 January 2007 / Accepted: 16 October 2008
Published online: 16 December 2008 –

Abstract: We present a connection between the BFV-complex (abbreviation for Batalin-Fradkin-Vilkovisky complex) and the strong homotopy Lie algebroid associated to a coisotropic submanifold of a Poisson manifold. We prove that the latter structure can be derived from the BFV-complex by means of homotopy transfer along contractions. Consequently the BFV-complex and the strong homotopy Lie algebroid structure are L_∞ quasi-isomorphic and control the same formal deformation problem.

However there is a gap between the non-formal information encoded in the BFV-complex and in the strong homotopy Lie algebroid respectively. We prove that there is a one-to-one correspondence between coisotropic submanifolds given by graphs of sections and equivalence classes of normalized Maurer-Cartan elemens of the BFV-complex. This does not hold if one uses the strong homotopy Lie algebroid instead.

Contents

1. Introduction

The geometry of coisotropic submanifolds inside Poisson manifolds is a very rich subject with connections to topics such as foliation theory, momentum maps, constrained systems and symplectic groupoids – see [W2] for instance. Recently a new algebraic structure called the "strong homotopy Lie algebroid" associated to such submanifolds has been investigated, e.g. [OP] in the symplectic setting or [CF] in the Poisson case. This structure is related to the deformation problem of a given coisotropic submanifold ([OP]) on the one hand and to the quantization of constrained systems ([CF]) on the other. Moreover it captures subtle properties of the foliation associated to a coisotropic submanifold ([Ki]).

The first main result of this paper is to reveal that the strong homotopy Lie algebroid is in some sense equivalent to a construction known as the BFV-complex – for a precise formulation see Theorem 5 in Subsect. 4.2. The BFV-complex originated from physical considerations concerning the quantization of field theories with so-called open gauge symmetries ([BF,BV]). It was given an interpretation in terms of homological algebra in [Sta2] and globalized to coisotropic submanifolds of arbitrary finite dimensional Poisson manifolds in [B and He].

Theorem 5 provides a connection between the BFV-complex and the strong homotopy Lie algebroid. In fact, we show that the two structures are isomorphic up to homotopy. In particular this implies (Corollary 4) that the formal deformation problem associated to both structures is equivalent. In [OP] this formal deformation problem was investigated in the setting of the strong homotopy Lie algebroid (in the symplectic case).

Remarkably there is a gap between the strong homotopy Lie algebroid and the BFV-complex in the non-formal regime: we present a simple example of a coisotropic submanifold inside a Poisson manifold where the strong homotopy Lie algebroid does not capture obstructions to deformations. However the BFV-complex always does, see Theorem 6 in Subsect. 5.2 for the precise statement. Hence the BFV-complex is able to capture non-formal aspects of the geometry of coisotropic submanifolds. This is also supported by the example considered in Subsect. 5.3 where the treatment using the BFV-complex reproduces a criterion for finding coisotropic submanifolds which was derived in [Z].

The paper is organized as follows: Section 2 collects known facts concerning algebraic and geometric structures that are used in the main body of the paper. In Sect. 3 we present the global construction of the BFV-complex. We mainly follow [Sta2,B and He] there. The only original part is the conceptual construction of the global BFV-bracket (see Subsect. 3.2). Section 4 introduces the strong homotopy Lie algebroid and connects it to the BFV-complex (Theorem 5). In Sect. 5 we establish a link between the BFV-complex and the geometry of coisotropic sections (Theorem 6) and give an example to demonstrate that this link does not exist if one considers the strong homotopy Lie algebroid instead. In the Appendix we give details on the homotopy transfer along contraction data which is one of our main tools. The material there is well known to the experts.

2. Preliminaries

For the convenience of the reader and in order to fix conventions we recall some basic definitions and facts concerning L_∞-algebras (Subsect. 2.1), the derived brackets formalism (Subsect. 2.2), homotopy transfer of L_∞-algebras along contraction data (Subsect. 2.3), smooth graded manifolds (Subsect. 2.4) and Poisson geometry (Subsect. 2.5). Readers familiar with these topics might skip this section.

2.1. L_∞-algebras. Let V be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$ (or any other field of characteristic 0); i.e., V is a collection $(V_i)_{i\in\mathbb{Z}}$ of vector spaces V_i over $\mathbb{R}$. Homogeneous elements of V of degree $i \in \mathbb{Z}$ are the elements of V_i. We denote the degree of a homogeneous element $x \in V$ by $|x|$. A morphism $f : V \to W$ of graded vector spaces is a collection $(f_i : V_i \to W_i)_{i\in\mathbb{Z}}$ of linear maps. The n^{th} suspension functor $[n]$ from the category of graded vector spaces to itself is defined as follows: given a graded vector space V, $V[n]$ denotes the graded vector space given by the collection $V[n]_i := V_{n+i}$. The n^{th} suspension of a morphism $f : V \to W$ of graded vector spaces is given by the collection $(f[n]_i := f_{n+i} : V_{n+i} \to W_{n+i})_{i\in\mathbb{Z}}$.

One can consider the tensor algebra $T(V)$ associated to a graded vector space V which is a graded vector space with components

$$T(V)_m := \bigoplus_{k\geq 0} \bigoplus_{j_1+\cdots+j_k=m} V_{j_1} \otimes \cdots \otimes V_{j_k}.$$

$T(V)$ naturally carries the structure of a cofree coconnected coassociative coalgebra given by the deconcatenation coproduct:

$$\Delta(x_1 \otimes \cdots \otimes x_n) := \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

There are two natural representations of the symmetric group Σ_n on $V^{\otimes n}$: the even one which is defined by multiplication with the sign $(-1)^{|a||b|}$ for the transposition interchanging homogeneous a and b in V and the odd one by multiplication with the sign $-(-1)^{|a||b|}$ respectively. These two actions naturally extend to $T(V)$. The fix point set of the first action on $T(V)$ is denoted by $S(V)$ and called the graded symmetric algebra of V while the fix point set of the latter action is denoted by $\Lambda(V)$ and called the graded skew–symmetric algebra of V. The graded symmetric algebra $S(V)$ inherits a coalgebra structure from $T(V)$ which is cofree coconnected coassociative and graded cocommutative.

Let V be a graded vector space together with a family of linear maps

$$(m^n : S^n(V) \to V[1])_{n\in\mathbb{N}}.$$

Given such a family one defines the associated family of Jacobiators

$$(J^n\colon S^n(V) \to V[2])_{n\geq 1}$$

by

$$J^n(x_1\cdots x_n) := \\ = \sum_{r+s=n}\ \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma)\, m^{s+1}(m^r(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)}),$$

where $\operatorname{sign}(\cdot)$ is the Koszul sign, i.e., the one induced from the natural even representation of Σ_n on $T^n(V)$, and (r,s)-shuffles are permutations σ of $\{1,\ldots,n\}$ such that $\sigma(1)<\cdots<\sigma(r)$ and $\sigma(r+1)<\cdots<\sigma(n)$.

Definition 1. *A family of maps $(m^n\colon S^n(V)\to V[1])_{n\in\mathbb{N}}$ defines the structure of an $L_\infty[1]$-algebra on the graded vector space V whenever the associated family of Jacobiators vanishes identically.*

This definition is the one given in [V]. We remark that this definition deviates from the classical notion of L_∞-algebras (see [LSt] for instance) in two points. First it makes use of the graded symmetric algebra over V instead of the graded skew–symmetric one. The transition between these two settings uses the so-called décalage-isomorphism

$$\begin{array}{rrcl} \operatorname{dec}^n\colon & S^n(V) & \to & \Lambda^n(V[-1])[n] \\ & x_1\cdots x_n & \mapsto & (-1)^{\sum_{i=1}^n (n-i)(|x_i|)} x_1\wedge\cdots\wedge x_n. \end{array}$$

The connection between $L_\infty[1]$-algebras and L_∞-algebras is easy:

Lemma 1. *Let W be a graded vector space. There is a one-to-one correspondence between $L_\infty[1]$-algebra structures on $W[1]$ and L_∞-algebra structures on W.*

More important is the fact that we also allow a map $m_0:\mathbb{R}\to V[1]$ as part of the structure given by an $L_\infty[1]$-algebra. This piece can be interpreted as an element of V_1. In the traditional definition m_0 is assumed to vanish. Relying on a widespread terminology, we call structures with $m_0=0$ "flat". Observe that in the case of a flat $L_\infty[1]$-algebra m_1 is a coboundary operator. Moreover $L_\infty[1]$-algebras with $m_k=0$ for all $k\neq 1,2$ correspond exactly to differential graded Lie algebras under the décalage-isomorphism:

Definition 2. *A* **graded Lie algebra** $(\mathfrak{h},[-,-])$ *is a graded vector space $\mathfrak{h}$ equipped with a linear map $[-,-]\colon\mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{h}$ satisfying the following conditions:*

- *graded skew-symmetry: $[x,y]=-(-1)^{|x||y|}[y,x]$ and*
- *graded Jacobi identity: $[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]]$,*

for all homogeneous $x\in\mathfrak{h}_{|x|}$, $y\in\mathfrak{h}_{|y|}$ and $z\in\mathfrak{h}$.

A differential graded Lie algebra is a triple $(\mathfrak{h},d,[-,-])$, where $(\mathfrak{h},[-,-])$ is a graded Lie algebra and d is a linear map of degree $+1$ such that $d\circ d=0$ and $d[x,y]=[dx,y]+(-1)^{|x|}[x,dy]$ holds for all $x\in\mathfrak{h}_{|x|}$ and $y\in\mathfrak{h}$.

If one goes from the category of graded vector spaces to the category of graded commutative associative algebras, the reasonable replacement of the notion of a (differential) graded Lie algebra is that of a (differential) graded Poisson algebra:

Definition 3. *A* **graded Poisson algebra** *is a triple* $(A, \cdot, [-,-])$*, where* $(A, \cdot)$ *is a graded commutative associative algebra and* $(A, [-,-])$ *is a graded Lie algebra such that* $[x, y \cdot z] = [x, y] \cdot z + (-1)^{|x||y|} y \cdot [x, z]$ *holds for* $x \in A_{|x|}$*,* $y \in A_{|y|}$ *and* $z \in A$.

A **differential graded Poisson algebra** *is a quadruple* $(A, d, \cdot, [-,-])$ *where* $(A, \cdot, [-,-])$ *is a graded Poisson algebra,* $(A, d, [-,-])$ *is a differential graded Lie algebra and* $d(x \cdot y) = dx \cdot y + (-1)^{|x|} x \cdot dy$ *holds for all* $x \in A_{|x|}$ *and* $y \in A$.

We briefly review a description of $L_\infty[1]$-algebras, equivalent to the one given in Definition 1, which goes back to Stasheff [Sta1]. We remarked before that the graded commutative algebra $S(V)$ associated to a graded vector space V is a cofree coconnected graded cocommutative coassociative coalgebra with respect to the coproduct Δ inherited from $T(V)$. A linear map $Q\colon S(V) \to S(V)$ that satisfies $\Delta \circ Q = (Q \otimes \mathrm{id} + \mathrm{id} \otimes Q) \circ \Delta$ is called a *coderivation* of $S(V)$. By cofreeness of the coproduct Δ it follows that every linear map from $S(V)$ to V can be extended to a coderivation of $S(V)$ and that every coderivation Q is uniquely determined by $\mathrm{pr} \circ Q$, where $\mathrm{pr}\colon S(V) \to V$ is the natural projection. So there is a one-to-one correspondence between families of linear maps $(m^n : S^n(V) \to V[1])_{n \in \mathbb{N}}$ and coderivations of $S(V)$ of degree 1. Moreover, the graded commutator equips $\oplus_{k \in \mathbb{Z}} \mathrm{Hom}(S(V), S(V)[k])[-k]$ with the structure of a graded Lie algebra and this Lie bracket restricts to the subspace of coderivations of $S(V)$. Odd coderivations Q that satisfy $[Q, Q] = 0$ are in one-to-one correspondence with families of maps whose associated Jacobiators vanish identically. Consequently, Maurer-Cartan elements of the space of coderivations of $S(V)$ correspond exactly to $L_\infty[1]$-algebra structures on V. Since $Q \circ Q = \frac{1}{2}[Q, Q] = 0$, Maurer–Cartan elements of the space of coderivations are exactly the codifferentials of $S(V)$.

We remark that the approach to $L_\infty[1]$-algebras outlined above makes the notion of $L_\infty[1]$-morphisms especially transparent: these are just coalgebra morphisms that are chain maps between the graded symmetric algebras equipped with the codifferentials that define the $L_\infty[1]$-algebra structures. There are two special kinds of $L_\infty[1]$-morphisms. As usual $L_\infty[1]$-isomorphisms are $L_\infty[1]$-morphisms with an inverse. Moreover there is the notion of $L_\infty[1]$ quasi-isomorphisms, i.e. those $L_\infty[1]$-morphisms which admit "inverses up to homotopy": consider an $L_\infty[1]$-morphism between flat $L_\infty[1]$-algebras, hence the unary structure maps are coboundary operators. The given $L_\infty[1]$-morphism also has a unary component which is a chain map for these coboundary operators. Consequently this map induces a map between the cohomologies. An $L_\infty[1]$ quasi-isomorphism is an $L_\infty[1]$-morphism between flat L_∞-algebras such that this induced map between cohomologies is an isomorphism. The notions of L_∞-morphisms, isomorphisms and quasi-isomorphisms are obtained from the corresponding notions in the category of $L_\infty[1]$-algebras using the identification under the décalage-isomorphism.

Associated to every L_∞-algebra structure $(m^n : \bigwedge^n(V) \to V[2-n])_{n \in \mathbb{N}}$ on a graded vector space V is a subset of V_1 given by the zero set of the so-called *MC-equation* (MC stands for Maurer-Cartan from now on) which reads

$$\sum_{n \geq 0} \frac{1}{n!} m^n(\mu \otimes \cdots \otimes \mu) = 0.$$

Elements of V_1 satisfying this equation are called *MC-elements*. We denote the set of all these elements by $MC(V)$. It is well-known that there is a natural action of V_0 on V by inner derivations. Integrating these one obtains a subgroup $Inn(V)$ of the automorphism group $Aut(V)$ of the L_∞-algebra V. There is an induced action on $MC(V)$. We will give a complete definition of the action of V_0 on $MC(V)$ for V being the BFV-complex in Subsect. 5.2.

2.2. Derived brackets formalism. We describe the derived brackets formalism essentially following [V].

Definition 4. *We call the triple* $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ *a* **V-algebra** *(V for Voronov) if* $(\mathfrak{h}, [\cdot, \cdot])$ *is a graded Lie algebra,* $\mathfrak{a}$ *is an abelian Lie subalgebra of* $\mathfrak{h}$ *– i.e.* $\mathfrak{a}$ *is a graded vector subspace of* $\mathfrak{h}$ *and* $[\mathfrak{a}, \mathfrak{a}] = 0$ *– and* $\Pi_{\mathfrak{a}} \colon \mathfrak{h} \to \mathfrak{a}$ *is a projection such that*

$$\Pi_{\mathfrak{a}}[x, y] = \Pi_{\mathfrak{a}}[\Pi_{\mathfrak{a}} x, y] + \Pi_{\mathfrak{a}}[x, \Pi_{\mathfrak{a}} y] \tag{1}$$

holds for every $x, y \in \mathfrak{h}$.

Instead of condition (1) one can require that $\mathfrak{h}$ splits into $\mathfrak{a} \oplus \mathfrak{p}$ as a graded vector space, where $\mathfrak{p}$ is also a graded Lie subalgebra of $\mathfrak{h}$. In terms of the projection, $\mathfrak{p}$ is given by the kernel of $\Pi_{\mathfrak{a}}$.

Let $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ be a V-algebra and pick an element $P \in \mathfrak{h}$ of degree +1. One can define the multilinear maps on $\mathfrak{a}$

$$\begin{array}{rccc} D_P^n \colon & \mathfrak{a}^{\otimes n} & \to & \mathfrak{a}[1] \\ & x_1 \otimes \cdots \otimes x_n & \mapsto & \Pi_{\mathfrak{a}}[[\ldots[[P, x_1], x_2], \ldots], x_n] \end{array} \tag{2}$$

for every $n \geq 0$. These maps are called the *higher derived brackets* associated to P. It is easy to check that all these maps are graded commutative, namely:

$$\begin{aligned} & D_P^n(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n) \\ & \quad = (-1)^{|x_i||x_{i+1}|} D_P^n(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n) \end{aligned}$$

for every $1 \leq i \leq n-1$. We restrict the higher derived brackets constructed from P to the symmetric algebra $S(\mathfrak{a})$ and obtain a family of maps $(D_P^n \colon S^n(\mathfrak{a}) \to \mathfrak{a}[1])_{n \in \mathbb{N}}$.

In [V] it is proven that the Jacobiators of the higher derived brackets $(D_P^n \colon S^n(\mathfrak{a}) \to \mathfrak{a}[1])_{n \in \mathbb{N}}$ associated to P are given by the higher derived brackets associated to $\frac{1}{2}[P, P]$:

$$J_{D_P}^n = D_{\frac{1}{2}[P,P]}^n.$$

It follows that all Jacobiators vanish identically if we assume that $[P, P] = 0$ holds. Elements P of degree 1 that satisfy $[P, P] = 0$ are exactly the MC-elements of the graded Lie algebra $\mathfrak{h}$. Hence one obtains:

Theorem 1. *Let* $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ *be a V-algebra and* P *a MC-element of* $(\mathfrak{h}, [-, -])$. *Then the family of higher derived brackets associated to* P

$$(D_P^n \colon S^n(\mathfrak{a}) \to \mathfrak{a}[1])_{n \in \mathbb{N}},$$

equips $\mathfrak{a}$ *with the structure of an* $L_\infty[1]$*-algebra (see Definition 1).*

2.3. Homotopy transfer. We describe a way to transfer L_∞-algebras along contractions. Since we are not primarily interested in this transfer-procedure for its own sake but rather as a tool, we will not state the results of this subsection in the largest possible generality.

The two most serious restrictions are that we will assume 1) that the L_∞-algebra we desire to transfer is a differential graded Lie algebra and 2) that the target of the transfer is the cohomology. We remark that a straightforward generalization of the procedure we are going to present 1) works for arbitrary L_∞-algebras and 2) more general subcomplexes than the cohomology can be treated. See [GL] for instance.

The situation is as follows: Let X be a graded vector space and d a coboundary operator on X (i.e. $d : X \to X[1]$ and $d \circ d = 0$). We denote the cohomology $H(X, d)$ by H. Assume that there are linear maps

- $h : X \to X[-1]$,
- $pr : X \to H$ surjective, and
- $i : H \to X$ injective,

such that the following conditions hold:

- i and pr are chain maps (i.e. $d \circ i = 0$ and $pr \circ d = 0$),
- $pr \circ i = id_H$,
- $id_X - i \circ pr = d \circ h + h \circ d$, and
- $h \circ h = 0$, $h \circ i = 0$ and $pr \circ h = 0$ (sideconditions).

The tupel (X, d, h, i, pr) is called *contraction data* and can be encoded in the following diagram:

$$(H, 0) \underset{pr}{\overset{i}{\rightleftarrows}} (X, d) \ , h.$$

Theorem 2. *Let (X, d, h, i, pr) be a graded vector space equipped with contraction data and a finite compatible filtration, i.e. a collection of graded vector subspaces*

$$X = \mathcal{F}_0 X \supseteq \mathcal{F}_1 X \supseteq \cdots \supseteq \mathcal{F}_n X \supseteq \mathcal{F}_{(n+1)} X \supseteq \cdots$$

such that $\mathcal{F}_N X = \{0\}$ for N large enough, satisfying

- $d(\mathcal{F}_k X) \subset \mathcal{F}_k X$ *for all* $k \geq 0$ *and*
- $h(\mathcal{F}_k X) \subset \mathcal{F}_k X$ *for all* $k \geq 0$.

Furthermore suppose X is equipped with the structure of a differential graded Lie algebra $(X, D, [-, -])$ such that

- $(D - d)(\mathcal{F}_k X) \subset \mathcal{F}_{(k+1)} X$.

Then the cohomology H of (X, d) is naturally equipped with the structure of a (flat) L_∞-algebra and there is a well-defined L_∞-morphism $\hat{i} : H \rightsquigarrow X$.

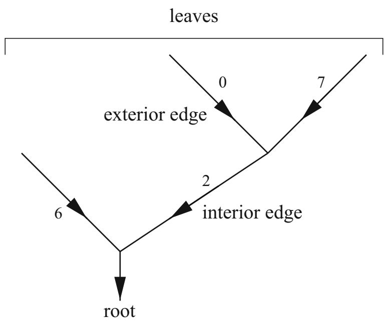
In all the cases where we apply Theorem 2 it is easy to check that the L_∞-morphism described in Lemma 3 is in fact an L_∞ quasi-isomorphism.

The conceptual proof of Theorem 2 is straightforward and can be found in [GL] for instance. One makes use of the interpretation of the L_∞-algebra structure on X as a codifferential Q on $S(X[1])$ and uses transfer formulae for Q to obtain a codifferential $\mathcal{Q}$ on $S(H[1])$, i.e. a L_∞-algebra structure on H. Moreover there are well-known formulas for $\hat{i}$.

Although Theorem 2 establishes the existence of a transfer-procedure along contraction data, we need a more concrete description of the induced L_∞-algebra and of the L_∞ quasi-isomorphism between H and X. Such a description was first given in the setting of A_∞-algebras: in [Me] inductive formulae were presented for the structure maps of the induced structure and in [KS] an interpretation in terms of Feynman diagrams was provided. Similar descriptions are known to hold for the transfer of L_∞-algebras as well, although we need a slight generalization of the setting presented in [Me and KS] since we allow the coboundary operator D to deviate from d.

We present the description of the transfer along contraction data using diagrams. Since we do not claim any originality on the material which is well-known to the experts, we only state the results. The interested reader can find the proofs in the Appendix.

An *oriented decorated tree* T is a finite connected graph without loops of any kind that only consists of directed edges and trivalent interior vertices with two incoming edges and one outgoing one. There are two kinds of exterior vertices: ones with an outgoing edge – we call these leaves – and exactly one with an incoming edge that we call the root. The orientation is given by an association of two numbers to any pair of edges with the same vertex as their target that tells us which of the two edges is the "right" and which is the "left" one. The decoration is an assignment of a non-negative integer to each edge.



The edge of the diagram with consists of only one leaf which is connected to the root must be decorated by a positive integer. Clearly we have a decomposition

$$\mathbb{T} = \bigsqcup_{n \geq 1} \mathbb{T}(n),$$

where $\mathbb{T}(n)$ denotes the set of trees with exactly n leaves. We will denote the set of unoriented decorated trees by $[\mathbb{T}]$. There is a natural projection

$$[\cdot] : \mathbb{T} \to [\mathbb{T}]$$

that respects the decomposition of $\mathbb{T}$ and that of $[\mathbb{T}]$:

$$[\mathbb{T}] = \bigsqcup_{n \geq 1} [\mathbb{T}](n) = \bigsqcup_{n \geq 1} [\mathbb{T}(n)].$$

We define $|Aut(T)|$ for T an oriented decorated tree to be the cardinality of the group of automorphisms of the underlying unoriented decorated tree.

Consider X equipped with contraction data (X, d, h, i, pr) and the structure of a differential graded Lie algebra $(X, D, [-,-])$ satisfying all conditions stated in Theorem 2. Then one assigns to any tree $T \in \mathbb{T}(k)$ a map

$$m_T : (H[1])^{\otimes k} \to H[2]$$

as follows: Using the décalage-isomorphism we equip $X[1]$ with the structure of an $L_\infty[1]$-algebra with structure maps μ^1 and μ^2 (corresponding to D and $[-,-]$ respectively). We write $\mu^1 = d + \mu^1_\Delta$. Next we put a μ^2 at each interior vertex of $T \in \mathbb{T}(k)$ and a number of μ^1_Δs at every edge – the number of μ^1_Δs is given by the number decorating the edge under consideration. Between any two consecutive operations one puts $-h$. Finally one places i at the leaves and pr at the root. The orientation of the tree induces a numbering of the leaves of T and applying all these maps in the order given by the orientation of the tree yields the map m_T.

It is easy to check that the "symmetrization"

$$\sum_{\sigma \in \Sigma_k} \frac{1}{|Aut(T)|} \sigma^*(m_T)$$

does not depend on the specific choice of the orientation of T.

Hence we get a map

$$\hat{m} : [\mathbb{T}] \to Hom(S(H[1]), H[2]),$$

and consequently

$$\nu^k := \sum_{[T] \in [\mathbb{T}](k)} \hat{m}([T])$$

is well-defined.

Lemma 2. *The sequence of maps* $(\nu^k : S^k(H[1]) \to H[2])_{k \geq 1}$ *defines the structure of an* $L_\infty[1]$*-algebra on* $H[1]$.

See the Appendix for a proof of this statement.

The $L_\infty[1]$-morphism $\hat{i} : H[1] \rightsquigarrow X[1]$ is also given in terms of oriented decorated trees. This time we associate the following map:

$$n_T : H[1]^{\otimes k} \to X[1]$$

to a tree T in $\mathbb{T}(k)$: again place μ^2 at all interior vertices, l copies of μ^1_Δ at edges decorated by l and between two consecutive operations of this kind place $-h$. As before put i at the leaves. The only difference is that we put a $-h$ at the root instead of pr. Again it is straightforward to check that the "symmetrization"

$$\sum_{\sigma \in \Sigma_k} \frac{1}{|Aut(T)|} \sigma^*(n_T)$$

does not depend on the choice of orientation of T and we obtain a map

$$\hat{n} : [\mathbb{T}] \to Hom(S(H[1]), X[1]).$$

One defines a family of maps

$$\lambda^k := \sum_{[T]\in[\mathbb{T}](k)} \hat{n}([T])$$

that satisfies

Lemma 3. *The sequence of maps* $(\lambda^k : S^k(H[1]) \to X[1])_{k\geq 1}$ *defines an* $L_\infty[1]$-*morphism between* $(H[1], \nu^1, \nu^2, \dots)$ *and* $(X[1], \mu^1, \mu^2)$.

The interested reader can find a proof of this statement in the Appendix.

2.4. Smooth graded manifolds.

Definition 5. *Let M be a smooth finite dimensional manifold. A* **(bounded) graded vector bundle** *over M is a collection $E_\bullet = (E_i)_{i\in\mathbb{Z}}$ of finite rank vector bundles over M such that $E_k = \{0\}$ for k smaller than some k_{min} or larger than some k_{max}. Since we only consider bounded graded vector bundles we will drop the adjective bounded from now on.*

The **algebra of smooth functions** *on a graded vector bundle $E_\bullet$ is the graded commutative associative algebra*

$$\mathcal{C}^\infty(E_\bullet) := \Gamma(\otimes_{k\in\mathbb{Z}}\mathcal{T}^{-k\bullet}(E_k^*)),$$

where $\mathcal{T}^{-k\bullet}(E_k^*)$ is $\bigwedge^{-k\bullet}(E_k^*)$ for k odd and $S^{-k\bullet}(E_k^*)$ for k even. The symbol $\otimes$ refers to the completed tensor product over $\mathcal{C}^\infty(M)$. Moreover the algebraic structure on the tensor product of two graded associative algebras is declared to be $(a\otimes x)\cdot(b\otimes y) := (-1)^{|x||b|}(a\cdot b)\otimes(x\cdot y)$.

A **morphism** *between two graded vector bundles $E_\bullet$ and $F_\bullet$ is a morphism of unital graded commutative associative algebras from $\mathcal{C}^\infty(F_\bullet)$ to $\mathcal{C}^\infty(E_\bullet)$.*

We define the n^{th} suspension operator $[n]$ on smooth graded vector bundles by $E_\bullet[n] := (E_{i+n})_{i\in\mathbb{Z}}$.

Definition 6. *A* **smooth graded manifold** *$\mathcal{M}$ is a unital graded commutative associative algebra $\mathcal{A}_M$ that is isomorphic to $\mathcal{C}^\infty(E_\bullet)$ for some graded vector bundle $E_\bullet$. We define $\mathcal{C}^\infty(\mathcal{M}) := \mathcal{A}_M$.*

A **morphism** *between two smooth graded manifolds $\mathcal{M}$ and $\mathcal{N}$ is a morphism of unital graded commutative algebras from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\mathcal{M})$.*

We remark that a specific isomorphism between $\mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{C}^\infty(E_\bullet)$ is not part of the data that define the smooth graded manifold $\mathcal{M}$.

Let $\mathcal{M}$ be a smooth graded manifold and let $\mathcal{X}(\mathcal{M})$ be the vector space of graded derivations of $\mathcal{C}^\infty(\mathcal{M})$, i.e. $\phi \in \mathcal{X}_k(\mathcal{M})$ iff

$$\phi : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[k]$$

satisfies $\phi(a\cdot b) = \phi(a)\cdot b + (-1)^{k|a|}a\cdot\phi(b)$ for homogeneous a and b in $\mathcal{C}^\infty(\mathcal{M})$.

Definition 7. *Let $\mathcal{M}$ be a smooth graded manifold. The algebra of* **multivector fields** *on $\mathcal{M}$ is the graded commutative associative algebra*

$$\mathcal{V}(\mathcal{M}) := S_{\mathcal{C}^\infty(\mathcal{M})}(\mathcal{X}(\mathcal{M})[-1]),$$

i.e. the graded symmetric algebra generated by $\mathcal{X}(\mathcal{M})[-1]$ as a graded module over $\mathcal{C}^\infty(\mathcal{M})$.

Let $\phi, \psi \in \mathcal{X}(\mathcal{M})$ be homogeneous elements of degree $|\phi|$ and $|\psi|$ respectively. Then

$$[\phi, \psi] := \phi \circ \psi - (-1)^{|\phi||\psi|}\psi \circ \phi$$

defines the structure of a graded Lie algebra on $\mathcal{X}(\mathcal{M})$. This bracket can be extended to a graded Lie algebra bracket $[-,-]_{SN}$ (SN stands for Schouten-Nijenhuis) on $\mathcal{V}(\mathcal{M})[1]$ by imposing the condition that $[-,-]_{SN}$ is a graded biderivation of $\mathcal{V}(\mathcal{M})$.

Assume that the smooth graded manifold $\mathcal{M}$ is represented by the graded vector bundle $E_\bullet \to M$. Using connections on the components of $E_\bullet$ one sees that there is an isomorphism between $\mathcal{V}(\mathcal{M})$ and $\mathcal{C}^\infty(T^*[1]M \oplus E_\bullet \oplus E_\bullet^*[1])$, where $E_\bullet^*$ refers to the graded vector bundle $(E_{-i}^*)_{i\in\mathbb{Z}}$. Hence:

Lemma 4. *Let $\mathcal{M}$ be a smooth graded manifold. Then the graded commutative algebra of multivector fields $\mathcal{V}(\mathcal{M})$ on $\mathcal{M}$ defines a smooth graded manifold.*

Let $Z \in \mathcal{V}(\mathcal{M})$ be a bivector field (i.e. an element of $S^2_{\mathcal{C}^\infty(\mathcal{M})}(\mathcal{X}(\mathcal{M})[-1])$) on $\mathcal{M}$ of total degree 0. The algebra $\mathcal{C}^\infty(\mathcal{M})[1]$ is an abelian Lie subalgebra of $(\mathcal{V}(\mathcal{M})[1], [-,-]_{SN})$ hence we can construct the derived brackets (D_Z^n) associated to Z, see Subsect. 2.2. The only possible non-vanishing term is D_Z^2. Using the décalage-isomorphism we obtain a map $\bigwedge^2(\mathcal{C}^\infty(\mathcal{M})) \to \mathcal{C}^\infty(\mathcal{M})$ which we denote by $[-,-]_Z$. According to Theorem 1 in Subsect. 2.2, $[-,-]_Z$ equips $\mathcal{C}^\infty(\mathcal{M})$ with the structure of a graded Lie algebra if Z satisfies $[Z,Z]_{SN} = 0$. It can be checked in this case that $(\mathcal{C}^\infty(\mathcal{M}), [-,-]_Z)$ is a graded Poisson algebra.

2.5. Poisson geometry. Let M be a smooth finite dimensional manifold. In Subsect. 2.4 the Schouten-Nijenhuis bracket $[-,-]_{SN}$ was introduced: it equips $\mathcal{V}(M)[1]$ with the structure of a graded Lie algebra. A Poisson bivector field Π on M is a MC-element of $(\mathcal{V}(M)[1], [-,-]_{SN})$, i.e. Π is a bivector field satisfying $[\Pi, \Pi]_{SN} = 0$.

Associated to any Poisson bivector field Π on M there is a vector bundle morphism $\Pi^\# : T^*M \to TM$ given by contraction. Denote the natural pairing between TM and T^*M by $<-,->$. The bracket on $\mathcal{C}^\infty(M)$ defined by $[f,g]_\Pi := <\Pi^\#(df), dg>$ is $\mathbb{R}$-bilinear, skew-symmetric, satisfies the Jacobi-identity and is a biderivation for the multiplication on $\mathcal{C}^\infty(M)$. Hence $(\mathcal{C}^\infty(M), [-,-]_\Pi)$ is a Poisson algebra.

Every Poisson manifold comes along with a singular foliation $\mathcal{F}_\Pi$, given by

$$\Pi^\#(T^*M) \hookrightarrow TM.$$

Locally this foliation is spanned by elements of the form $\Pi^\#(df)$ for $f \in \mathcal{C}^\infty(M)$. The identity

$$[\Pi^\#(df), \Pi^\#(dg)]_{SN} = \Pi\#(d[f,g]_\Pi)$$

is satisfied which implies that $\mathcal{F}_\Pi$ is involutive. By a generalization of the classical theorem of Frobenius due to Stefan and Sussman (see [Ste, Su]) the integrability of $\mathcal{F}_\Pi$ follows. The integrating leaves all carry a natural symplectic structures induced from Π.

There is another interesting structure associated to every Poisson manifold (M, Π). Consider the binary operation on $\Gamma(T^*M) = \Omega^1(M)$ given by

$$[\alpha, \beta]_K := \mathcal{L}_{\Pi^\#(\alpha)}(\beta) - \mathcal{L}_{\Pi^\#(\beta)}(\alpha) + d\Pi(\alpha, \beta)$$

called the Koszul bracket. One can check that it is a Lie bracket on $\Omega^1(M)$ and that the vector bundle morphism $\Pi^\# : T^*M \to TM$ induces a morphism of Lie algebras $(\Omega^1(M), [-,-]_K) \to (\mathcal{X}(M), [-,-]_{SN})$. Moreover the so-called Leibniz identity holds:

$$([\alpha, f\beta]_K) = f[\alpha, \beta]_K + \Pi^\#(\alpha)(f) \cdot \beta$$

for all $\alpha, \beta \in \Omega^1(M)$ and $f \in \mathcal{C}^\infty(M)$. The triple $(T^*M, [-,-]_K, \Pi^\#)$ is an example of a Lie algebroid over M. Associated to any Lie algebroid is a cocomplex, called the Lie algebroid cocomplex. In fact this cocomplex encodes exactly the same information as the original Lie algebroid data. In the case of the Lie algebroid $(T^*M, [-,-]_K, \Pi^\#)$ the Lie algebroid cocomplex is $(\mathcal{V}(M), [\Pi, -]_{SN})$.

Consider a submanifold S of M. The annihilator N^*S of TS is a natural subbundle of T^*M. This subbundle fits into a short exact sequence of vector bundles:

$$0 \longrightarrow N^*S \longrightarrow T^*_S M \longrightarrow T^*S \longrightarrow 0\,.$$

Definition 8. *A submanifold S of a smooth finite dimensional Poisson manifold (M, Π) is called* **coisotropic** *if the restriction of $\Pi^\#$ to N^*S has image in TS.*

Consequently any coisotropic submanifold S is equipped with a natural singular foliation $\mathcal{F}_S := \Pi^\#(N^*S)$ which is involutive. Involutivity of $\mathcal{F}_S$ follows from another equivalent characterization of coisotropic submanifolds: define the vanishing ideal of S by

$$\mathcal{I}_S := \{f \in \mathcal{C}^\infty(M) : f|_S = 0\}.$$

A submanifold S is coisotropic if and only if $\mathcal{I}_C$ is a Lie subalgebra of $(\mathcal{C}^\infty(M), [-,-]_\Pi)$. Observe that $\Pi^\#(N^*S)$ is locally spanned by $\Pi^\#(df)$ for $f \in \mathcal{I}_S$. For $f, g \in \mathcal{I}_S$ one has $[\Pi^\#(df), \Pi^\#(dg)]_{SN} = \Pi\#(d[f,g]_\Pi)$. Since $[f,g]_\Pi \in \mathcal{I}_S$ the foliation $\mathcal{F}_S$ is involutive. We denote the corresponding leaf space by $\underline{S} := S/\sim_{\mathcal{F}_S}$. This space is usually very ill-behaved (non-smooth, non-Hausdorff, etc.). In particular there might not be a meaningful way to define $\mathcal{C}^\infty(\underline{S})$ using the topological space $\underline{S}$. Instead one can define $\mathcal{C}^\infty(\underline{S})$ as the space of functions on S which are invariant under $\mathcal{F}_S$, i.e.

$$\mathcal{C}^\infty(\underline{S}) := \{f \in \mathcal{C}^\infty(S) : X(f) = 0 \text{ for all } X \in \Gamma(\mathcal{F}_S)\}.$$

This is a subalgebra of $\mathcal{C}^\infty(S)$.

Fix an embedding $\phi : NS \hookrightarrow M$ of the normal bundle of S into M. Via the identification of NS with an open neighbourhood of S in M the vector bundle NS inherits a Poisson bivector field Π_ϕ. Hence we can assume without loss of generality that M is the total space of a vector bundle $E \to S$. We will do so in the rest of the paper. Observe that under the above assumptions there is a natural isomorphism $E \cong NS$.

With the help of this assumption one sees that $\mathcal{C}^\infty(\underline{S})$ comes equipped with a Poisson bracket $[-,-]_{\underline{S}}$ inherited from (E, Π): the algebra $\mathcal{C}^\infty(S)$ is the quotient of $\mathcal{C}^\infty(E)$ by $\mathcal{I}_S$. There is a Lie algebra action of $(\mathcal{I}_S, [-,-]_\Pi)$ on the quotient. The algebra $\mathcal{C}^\infty(\underline{S})$ is given by the invariants under this action, i.e.

$$\mathcal{C}^\infty(\underline{S}) \cong (\mathcal{C}^\infty(E)/\mathcal{I}_S)^{\mathcal{I}_S}.$$

This algebra is isomorphic to the quotient of

$$\mathcal{N}(\mathcal{I}_S) := \{f \in \mathcal{C}^\infty(E) : [f, \mathcal{I}_S]_\Pi \subset \mathcal{I}_S\}$$

by $\mathcal{I}_S$. It is straightforward to check that the Poisson bracket on $\mathcal{C}^\infty(E)$ descends to this quotient.

The Lie algebroid structure $(T^*M, [-,-]_K, \Pi^\#)$ also restricts to coisotropic submanifolds: the bundle map $\Pi^\# : T^*E \to TE$ restricts to a bundle map $E^* \to TS$ by definition and the Koszul bracket can also be restricted to $\Gamma(E^*)$. The triple $(E^*, [-,-]_K, \Pi^\#|_{E^*})$ satisfies the same identities as $(T^*E, [-,-]_K, \Pi^\#)$ and hence is a Lie algebroid, see [W2] for details. The easiest way to describe this Lie algebroid over S is via its associated Lie algebroid cocomplex. Define a projection $pr : \mathcal{V}(M) \to \Gamma(\bigwedge E)$ as the unique algebra morphism extending the restriction $\mathcal{C}^\infty(E) \to \mathcal{C}^\infty(S)$ and $\mathcal{X}(E) = \Gamma(TE) \to \Gamma(T_S E) \to \Gamma(E)$. The graded algebra $\Gamma(\bigwedge E)$ is equipped with the differential given by

$$\partial_S(X) := pr([\Pi, \tilde{X}]_{SN}|_S),$$

where $\tilde{X}$ is any extension of $X \in \Gamma(\bigwedge E)$ to a multivector field on E. The cohomology of the cocomplex $(\Gamma(\bigwedge E), \partial_S)$ is called the *Lie algebroid cohomology* of S. It is well-known that

Lemma 5. *Let S be a coisotropic submanifold of a smooth finite dimensional Poisson manifold (M, Π). The algebra $\mathcal{C}^\infty(\underline{S})$ is isomorphic to the degree zero Lie algebroid cohomology $H^0(\bigwedge(NS), \partial_S)$.*

Moreover it is possible to show that the Lie algebroid differential ∂_S is independent of the embedding $NS \hookrightarrow M$ as is the Poisson bracket on $\mathcal{C}^\infty(\underline{S})$.

3. The BFV-Complex

Consider a finite rank vector bundle $E \to S$ that is equipped with a Poisson bivector field, i.e. $\Pi \in \mathcal{V}^2(E)$ satisfying $[\Pi, \Pi]_{SN} = 0$, such that S is a coisotropic submanifold. Let S be a coisotropic submanifold of E.

The aim of this section is to describe the construction of a homological resolution of the Poisson algebra $(\mathcal{C}^\infty(\underline{S}), [-,-]_{\underline{S}})$ (introduced in Subsect. 2.5) in terms of a differential graded Poisson algebra

$$(BFV(E, \Pi), D_{BFV}, [-,-]_{BFV}).$$

$BFV(E, \Pi)$ can be described as the space of smooth functions on some smooth graded manifold. The degree zero component of the cohomology $H(BFV(E, \Pi), D_{BFV})$ is isomorphic to $\mathcal{C}^\infty(\underline{S})$ and the induced bracket coincides with $[-,-]_{\underline{S}}$.

The basic ideas of the construction of $(BFV(E, \Pi), D_{BFV}, [-,-]_{BFV})$ were invented by Batalin, Fradkin and Vilkovisky ([BF, BV]) with applications to physics in mind. Later it was reinterpreted by Stasheff in terms of homological algebra ([Sta2]). The convenient globalization to the smooth setting was presented by Bordemann and Herbig ([B, He]). We essentially follow [Sta2, B and He] in this exposition. The only deviation will be a new conceptual approach to the Rothstein-bracket ([R]) and its extension to the Poisson setting ([He]) in terms of higher homotopy structures given in Sect. 3.2.

The construction of the BFV-complex relies on the following input data: 1) a choice of embedding of the normal bundle of S as a tubular neighourhood (in order to obtain an appropriate vector bundle $E \to S$, see Subsect. 2.5), 2) a connection on $E \to S$, and 3) a distinguished element $\Omega \in BFV(E, \Pi)$ satisfying $[\Omega, \Omega]_{BFV} = 0$. The dependence of the resulting differential graded Poisson algebra on these data will be clarified elsewhere ([Sch]).

3.1. The ghost/ghost-momentum bundle. Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold. Using the projection map of the vector bundle $E \to S$ we can pull back the graded vector bundle $E^*[1] \oplus E[-1] \to S$ to a graded vector bundle over E which we denote by $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$. The situation is summarized by the following Cartesian square:

$$\begin{tikzcd} \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \arrow[r] \arrow[d, "P"'] & E^*[1] \oplus E[-1] \arrow[d] \\ E \arrow[r] & S. \end{tikzcd}$$

We define $BFV(E, \Pi)$ to be the space of smooth functions on the graded manifold which is represented by the graded vector bundle $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ over E. In terms of sections one has $BFV(E, \Pi) = \Gamma(\bigwedge(\mathcal{E}) \otimes \bigwedge(\mathcal{E}^*))$. This algebra carries a bigrading given by

$$BFV^{(p,q)}(E, \Pi) := \Gamma(\wedge^p(\mathcal{E}) \otimes \wedge^q(\mathcal{E}^*)).$$

In physical terminology p/q is referred to as the *ghost degree/ghost-momentum degree* respectively. One defines

$$BFV^k(E, \Pi) := \bigoplus_{p-q=k} BFV^{(p,q)}(E, \Pi)$$

and calls k the *total degree* (in physical terminology this is the "ghost number"). There is yet another decomposition of $BFV(E, \Pi)$ that will be useful later: set $BFV_r(E, \Pi) := \Gamma(\bigwedge(\mathcal{E}) \otimes \bigwedge^r(\mathcal{E}^*))$. Moreover we define $BFV_{\geq r}(E, \Pi)$ to be the ideal generated by $BFV_r(E, \Pi)$.

The smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ comes equipped with a Poisson bivector field G given by the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$, i.e. it is defined to be the natural contraction on $\Gamma(\mathcal{E}) \otimes \Gamma(\mathcal{E}^*)$ and it extends uniquely to a graded skew-symmetric biderivation of $BFV(E, \Pi)$.

3.2. Lifting the Poisson bivector field. We want to equip $BFV(E, \Pi)$ with the structure of a graded Poisson algebra which essentially combines the Poisson bivector field Π on E and the Poisson bivector field G which encodes the natural fibre pairing between $\mathcal{E}^*[1]$ and $\mathcal{E}[-1]$.

First we lift Π from the base E to the graded vector bundle

$$\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \xrightarrow{P} E.$$

For this purpose we choose a connection ∇ on the vector bundle $E \to S$. This yields a connection on $E^*[1] \oplus E[-1]$. Pulling back this connection along $E \to S$ gives a connection on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ that is metric with respect to the natural fibre pairing. Fix such a connection on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to S$ and consider the horizontal lift with respect to that connection, i.e. we obtain a map $\iota_\nabla : \mathcal{X}(E) \hookrightarrow \mathcal{X}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$. Setting $\iota_\nabla(f) := f \circ P$ for $f \in \mathcal{C}^\infty(E)$ we can uniquely extend ι_∇ to a morphism of algebras

$$\iota_\nabla : \mathcal{V}(E) \hookrightarrow \mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]).$$

Since $\iota_\nabla[1]$ fails in general to be a morphism of graded Lie algebras, the horizontal lift $\iota_\nabla(\Pi)$ of the Poisson bivector field Π does not satisfy the MC-equation in $(\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])[1],[-,-]_{SN})$. The same is true for the sum $G+\iota_\nabla(\Pi)$, hence this bivector field does not define the structure of a graded Poisson algebra on $BFV(E,\Pi)$. We will show that an appropriate correction term $\triangle$ can be found such that $G+\iota_\nabla(\Pi)+\triangle$ is a MC-element. The existence of such a $\triangle$ is the straightforward consequence of the following proposition:

Proposition 1. *Let $\mathcal{E}$ be a finite rank vector bundle with connection ∇ over a finite dimensional smooth manifold E. Consider the smooth graded manifold $\mathcal{E}^*[1]\oplus\mathcal{E}[-1]\to E$ and denote the Poisson bivector field on it coming from the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ by G.*

Then there is an L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ between the graded Lie algebra $(\mathcal{V}(E)[1],[-,-]_{SN})$ and the differential graded Lie algebra $(\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])[1],[G,-]_{SN},[-,-]_{SN})$.

Observe that it is not assumed in the proposition that E is a vector bundle or that $\mathcal{E}\to E$ is a pull back bundle.

Proof. Consider the induced connection ∇ on $\mathcal{E}^*[1]\oplus\mathcal{E}[-1]\to E$ (by a slight abuse of notation we denote this connection by ∇ too). It is metric with respect to the natural fibre pairing. The algebra morphism $\iota_\nabla:\mathcal{V}(E)\hookrightarrow\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])$ (given by the horizontal lift) is a section of the natural projection

$$Pr:\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])\to\mathcal{V}(E).$$

Obviously $Pr\circ\iota_\nabla=id$ holds on $\mathcal{V}(E)$.

Consider the complexes $(\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1]),Q:=[G,-]_{SN})$ and $(\mathcal{V}(E),0)$. It is easy to check that Pr and ι_∇ are chain maps. Here it is crucial that the induced connection on $\mathcal{E}^*[1]\oplus\mathcal{E}[-1]$ is metric with respect to the natural fibre pairing. We will construct a homotopy

$$H_\nabla:=\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])\to\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])[-1]$$

such that $Q\circ H_\nabla+H_\nabla\circ Q=id-\iota_\nabla\circ Pr$, i.e. ι_∇ and Pr are inverses up to homotopy and it follows that Pr induces an isomorphism $H(\mathcal{V}(E^*[1]\oplus E[-1]),Q)\cong\mathcal{V}(M)$.

To construct an appropriate homotopy H_∇ we extend ι_∇ to an algebra isomorphism

$$\varphi_\nabla:\mathcal{A}:=\mathcal{C}^\infty(T^*[1]E\oplus\mathcal{E}^*[1]\oplus\mathcal{E}[-1]\oplus\mathcal{E}[0]\oplus\mathcal{E}^*[2])\to\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1]),$$

see Lemma 4 in Subsect. 2.4. Via this identification we equip $\mathcal{A}$ with a Gerstenhaber bracket $[-,-]_\nabla$ and a differential $\tilde{Q}:=\varphi_\nabla^{-1}\circ Q\circ\varphi_\nabla$. Define $\tilde{H}$ to be the sum of the pull-backs by the maps $-id_{\mathcal{E}^*[1]}[1]:\mathcal{E}^*[1]\to\mathcal{E}^*[2]$ and $-id_{\mathcal{E}[-1]}[1]:\mathcal{E}[-1]\to\mathcal{E}[0]$ on $\mathcal{A}$. It is straightforward to check that $\tilde{H}$ is a differential and that $(\tilde{Q}\circ\tilde{H}+\tilde{H}\circ\tilde{Q})(X)$ is equal to the total polynomial degree of X in all of the fibre components $\mathcal{E}^*[1],\mathcal{E}^*[2],\mathcal{E}[-1]$ and $\mathcal{E}[0]$. Normalising $\tilde{H}$ and using the identification φ_∇ leads to a homotopy H_∇ on $\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])$. It is straightforward to check that the side-conditions $H_\nabla\circ H_\nabla=0$, $H_\nabla\circ\iota_\nabla=0$ and $Pr\circ H_\nabla=0$ hold.

We summarize the situation in the following diagram:

$$(\mathcal{V}(E),0)\underset{Pr}{\overset{\iota_\nabla}{\rightleftarrows}}(\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1]),Q)\ ,H_\nabla.$$

According to Subsect. 2.3 these data can be used to perform homological transfer of L_∞-algebra structures along the contraction Pr. Starting with the differential graded Lie algebra $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], Q = [G, -]_{SN}, [-, -]_{SN})$ one constructs an L_∞ quasi-isomorphic L_∞-algebra structure on $\mathcal{V}(E)[1]$ (with zero differential) together with an L_∞ quasi-isomorphism $\mathcal{L}_\nabla$. The binary operation of this structure will simply be given by

$$Pr([\iota_\nabla(-), \iota_\nabla(-)]_{SN}) = [-, -]_{SN}.$$

All potential higher operations can be checked to vanish as follows: As described in 2.3 one considers all trivalent oriented trees. On the leaves (i.e. exterior vertices with edges oriented away from them) one places ι_∇, on each interior trivalent vertex one places $[-, -]_{SN}$, on the root (i.e. the unique exterior vertex with edge oriented towards it) one places Pr and on interior edges (those not connected to any leaf or to the root) one places $-H_\nabla$. Then one composes these maps in the order given by the orientation of the tree.

To prove that no higher order operations occur we introduce a decomposition of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$. By definition this is the space of multiderivations of the graded unital algebra $\mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$. The algebra of smooth functions is bigraded which induces a bigrading on its tensor algebra (just take the sum of the bidegrees of all tensor components) which in turn induces a bigrading on the space of multivector fields, i.e. an element of bidegree (m, n) is one that maps a tensor product of function of total bidegree (p, q) to a function of bidegree $(p+m, q+n)$. This bidegree is obviously bounded from above. We denote the ideal generated by the components of bidegree (M, N) with $M \geq m$ and $N \geq n$ by $\mathcal{V}^{(m,n)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$.

Consider a tree as above and forget about Pr at the root. One can inductively show that the corresponding operation maps tensor products of elements of $\mathcal{V}(E)$ to $\mathcal{V}^{(e-1,e-1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$, where e is the number of trivalent vertices of the tree. This relies on the following

Lemma 6. *Denote the curvature of the connection ∇ on $\mathcal{E} \to E$ by R_∇. We interpret R_∇ as an element of $\Omega^2(E, End(\mathcal{E})) = \Omega^2(E, \mathcal{E} \otimes \mathcal{E}^*)$. Then*

$$R_\nabla(-, -) = H_\nabla([\iota_\nabla(-), \iota_\nabla(-)]_{SN})$$

holds.

Proof of the lemma. The right-hand side of the claimed equality can be checked to be $\mathcal{C}^\infty(E)$-bilinear and multiplicative in both slots with respect to the algebra structure on $\mathcal{V}(E)$. Hence it is determined by its values on a pair of vector fields and can be interpreted as a two-form on E with values in a vector bundle. Consequently it is enough to prove the equality locally which is a straightforward computation in coordinate charts. □

So all operations vanish identically after applying Pr except for the case of the tree with only one trivalent edge (which corresponds to the binary operation $[-, -]_{SN}$). □

Corollary 1. *Let $\mathcal{E} \to E$ be a finite rank vector bundle with connection ∇ over a smooth finite dimensional Poisson manifold (E, Π). Consider the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ and denote the Poisson bivector field on it coming from the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ by G.*

Then there is a Poisson bivector field $\hat{\Pi}$ on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ such that $\hat{\Pi} = G + \iota_\nabla(\Pi) + \triangle$ for $\triangle \in \mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$.

Recall that $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is the ideal of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ generated by multiderivations which map a tensor product of functions of total bidegree (p, q) to a function of bidegree (P, Q) where $P > p$ and $Q > q$.

This corollary was originally proven by Rothstein in [R] for (N, Π) symplectic with the help of a concrete formula for $\hat{\Pi}$. Herbig showed that Rothstein's formula holds also in the Poisson case ([He]).

Proof. The general theory of L_∞-algebras implies that given two L_∞ quasi-isomorphic L_∞-algebras and a formal MC-element of one of these L_∞-algebras, one can construct a formal MC-element of the other one. We apply this to the Poisson bivector field Π seen as a MC-element in $(\mathcal{V}(E)[1], [-,-]_{SN})$ which is L_∞ quasi-isomorphic to $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]), [G, -]_{SN}, [-,-]_{SN})$ according to Proposition 1.

The unary operation from $\mathcal{V}(E)$ to $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is given by ι_∇. The higher structure maps of the L_∞-morphism between $\mathcal{V}(E)$ and $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ are given in terms of trivalent oriented trees. One places ι_∇ at leaves (i.e. exterior vertices with edges oriented away from them), $[-,-]_{SN}$ at trivalent interior vertices and the homotopy $-H_\nabla$ at all interior edges (all edges not connected to a leaf or root) and at the edge connected to the root (the unique exterior vertex with the edge oriented towards it). There is an estimate similar to the one in the proof of Proposition 1: the operation corresponding to a tree with e trivalent edges maps elements of $\mathcal{V}(E)$ to $\mathcal{V}^{(e,e)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$.

This implies 1) that we do not have to care about convergence since the filtration of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ by the ideals $\mathcal{V}^{(k,l)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is bounded from above, so only finitely many trees will contribute; and 2) by applying the L_∞ quasi-isomorphism to Π one obtains a Maurer-Cartan element of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, -]_{SN}, [-,-]_{SN})$ of the form $\iota_\nabla(\Pi) + \triangle$ with $\triangle \in \mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$. This is equivalent to the statement that $G + \iota_\nabla(\Pi) + \triangle$ is a Maurer-Cartan element of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [-,-]_{SN})$ of the desired form. $\square$

By definition such an element yields the structure of a graded Poisson algebra on $\mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]) =: BFV(E, \Pi)$:

Corollary 2. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold. Consider the associated ghost/ghost-momentum bundle $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \xrightarrow{P} E$ with the embedding $j : E \hookrightarrow \mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ as the zero section. The natural fibre pairing between $\mathcal{E}^*$ and $\mathcal{E}$ gives rise to a Poisson bivector field G.*

Then there is a graded Poisson bracket $[-,-]_{BFV}$ on $BFV(E, \Pi)$ such that:

(1) $[-,-]_\Pi = j^*([P^*(-), P^*(-)]_{BFV})$ *and*
(2) *denoting the projection $BFV^0(E, \Pi) \to BFV^{(0,0)}(E, \Pi)$ by $proj$ the composition*

$$BFV^{(1,0)}(E, \Pi) \otimes BFV^{(0,1)}(E, \Pi) \xrightarrow{[-,-]_{BFV}} BFV^0(E, \Pi)$$
$$\xrightarrow{proj} BFV^{(0,0)}(E, \Pi)$$

coincides with the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$.

3.3. The BFV-charge. Next we construct a differential D_{BFV} on $BFV(E, \Pi)$ with special properties.

Proposition 2. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is a coisotropic submanifold. Consider the graded Poisson algebra $BFV(E, \Pi) := (\mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]), [-, -]_{BFV})$ with a bracket as in Corollary 2.*

Then there is an element $\Omega \in BFV(E, \Pi)$ of degree $+1$ such that

(1) $[\Omega, \Omega]_{BFV} = 0$ *and*
(2) Ω *mod* $BFV_{\geq 1}(E, \Pi)$ *is given by the tautological section of* $\mathcal{E} \to E$.

Recall that $\mathcal{E}$ is the pullback bundle of $E \to S$ by $E \to S$ which admits a tautological section. By the inclusions

$$\Gamma(\mathcal{E}) \hookrightarrow \Gamma(\wedge(\mathcal{E})) \hookrightarrow \Gamma(\wedge(\mathcal{E}) \otimes \wedge(\mathcal{E}^*)) = BFV(E, \Pi)$$

the tautological section can be seen as an element of $BFV^{(1,0)}(E, \Pi)$ which we denote by Ω_0.

The proof we give is a slight adaptation of the arguments in [Sta2]:

Proof. It is convenient to work in local coordinates: fix local coordinates $(x^\beta)_{\beta=1,\ldots,s}$ on S, linear fibre coordinates $(y^j)_{j=1,\ldots,e}$ along E, $(c_j)_{j=1,\ldots,e}$ along $\mathcal{E}^*[1]$ and $(b^j)_{j=1,\ldots,e}$ along $\mathcal{E}[-1]$. In local coordinates the tautological section reads

$$\Omega_0 := \sum_{j=1}^{e} y^j c_j.$$

Since $[\Omega_0, \Omega_0]_G = 0$ – G is the Poisson bivector field given by the natural fibre pairing between $\mathcal{E}^*[1]$ and $\mathcal{E}[-1]$ – we obtain a differential

$$\delta := [\Omega_0, -]_G = \sum_{j=1}^{e} y^j \frac{\vec{\partial}}{\partial b^j}.$$

Claim. $H(BFV(E, \Pi), \delta) \cong \mathcal{C}^\infty(E^*[1]) = \Gamma(\bigwedge E)$. There are natural maps

$$i : E^*[1] \hookrightarrow \mathcal{E}^*[1] \oplus \mathcal{E}[1] \text{ and}$$
$$p : \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E^*[1].$$

Define $h : BFV(E, \Pi) \to BFV(E, \Pi)[-1]$ by setting

$$h(f_{j_1\ldots j_k}(x, y, c) b^{j_1} \cdots b^{j_k}) := \sum_{1 \leq \mu \leq e} b^\mu \left(\int_0^1 \frac{\partial f_{j_1\ldots j_k}}{\partial y^\mu}(x, t \cdot y, c) t^k dt \right) b^{j_1} \cdots b^{j_k}$$

which is globally well-defined. It is straightforward to check $i^* \circ \delta = 0$, $\delta \circ p^* = 0$, $h \circ h = 0$, $i^* \circ h = 0$, $h \circ p^* = 0$ and $\delta \circ h + h \circ \delta = id - p^* \circ i^*$. It follows that $i^* : BFV(E, \Pi) \to \mathcal{C}^\infty(E^*[1])$ induces an isomorphism on cohomology.

First note that $[\Omega_0, \Omega_0]_{BFV}$ mod $BFV_{\geq 1}(E, \Pi) = [\Omega_0, \Omega_0]_{\iota_\nabla(\Pi)} =: 2R_0$. Using the biderivation property of $[-, -]_{\iota_\nabla(\Pi)}$ one sees that

$$[\Omega_0, \Omega_0]_{\iota_\nabla(\Pi)} = [y^i, y^j]_{\iota_\nabla(\Pi)} c_i c_j + 2y^i [c_i, y^j]_{\iota_\nabla(\Pi)} c_j + y^i y^j [c_i, c_j]_{\iota_\nabla(\Pi)}.$$

Because $[y^i, y^j]_{\iota_\nabla(\Pi)}$ is equal to the pull back of $[y^i, y^j]_\Pi$ along the projection $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$, the condition that $[y^i, y^j]_\Pi$ is contained in the vanishing ideal $\mathcal{I}_S$ of S for arbitrary $i, j = 1, \ldots, e$ is equivalent to the condition that R_0 vanishes when evaluated on S. Hence the fact that R_0 vanishes along S is equivalent to the fact that S is coisotropic, see Subsect. 2.5.

Because of $\delta([\Omega_0, \Omega_0]_{\iota_\nabla(\Pi)}) = 0$ we obtain a cohomology class $[R_0]$ in $H(BFV(E, \Pi), \delta) \cong \mathcal{C}^\infty(N^*[1]S)$. Since the isomorphism between the two cohomologies is induced by setting the fibre coordinates $(y^j)_{j=1,\ldots,e}$ and $(b^j)_{j=1,\ldots,e}$ to zero one sees that $[R_0] = 0$. Hence $R_0 = -\delta(\Omega_1)$ for some $\Omega_1 \in BFV_1(E, \Pi)$. Consequently

$$\begin{aligned}[\Omega_0 + \Omega_1, \Omega_0 + \Omega_1]_{BFV} \bmod BFV_{\geq 1}(E, \Pi) &= [\Omega_0, \Omega_0]_{\iota_\nabla(\Pi)} + [\Omega_0, \Omega_1]_G \\ &= 2R_0 + \delta(\Omega_1) = 0.\end{aligned}$$

Claim. Given $k > 0$ and $\Omega(k) := \sum_{1 \leq i \leq k} \Omega_k$ with Ω_0 as above, $\Omega_i \in \Gamma(\bigwedge^{(i+1)}(\mathcal{E}) \otimes \bigwedge^i(\mathcal{E}^*))$ and

$$[\Omega(k), \Omega(k)]_{BFV} = 0 \bmod BFV_{\geq k}(E, \Pi),$$

there is an $\Omega_{k+1} \in BFV_{k+1}(E, \Pi)$ of total degree +1 such that $\Omega(k+1) := \Omega(k) + \Omega_{k+1}$ satisfies

$$[\Omega(k+1), \Omega(k+1)]_{BFV} = 0 \bmod BFV_{\geq (k+1)}(E, \Pi).$$

Set $2R_k := [\Omega(k), \Omega(k)]_{BFV} \bmod BFV_{\geq (k+1)}(E, \Pi)$, hence $R_k \in BFV_k(E, \Pi)$. By the graded Jacobi identity we know that $[\Omega(k), [\Omega(k), \Omega(k)]_{BFV}]_{BFV} = 0$. Moreover $[\Omega(k), \Omega(k)]_{BFV} = 2R_k \bmod BFV_{\geq k+1}(E, \Pi)$ implies that

$$\begin{aligned}0 = [\Omega(k), [\Omega(k), \Omega(k)]_{BFV}]_{BFV} &= [\Omega_0, 2R_k]_{BFV} \bmod BFV_{\geq k}(E, \Pi) \\ &= \delta(2R_k).\end{aligned}$$

So R_k is δ-closed and using $H(BFV(E, \Pi), \delta) \cong \mathcal{C}^\infty(E^*[1])$ we can conclude that there is an element $\Omega_{k+1} \in BFV_{k+1}(E, \Pi)$ of total degree +1 such that $R_k = -\delta(\Omega_{k+1})$. It is easy to check that this element satisfies the conditions of the claim.

After finitely many steps this procedure is finished thanks to the boundedness of the filtration $BFV_{\geq k}(E, \Pi)$. The (well-defined) element

$$\Omega := \sum_{k \geq 0} \Omega_k$$

satisfies properties 1) and 2) of the proposition by construction. □

Definition 9. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold. Assume (E, Π) is a Poisson manifold and S is a coisotropic submanifold.*

A differential graded Poisson algebra $(BFV(E, \Pi), D_{BFV} := [\Omega, -]_{BFV}, [-, -]_{BFV})$ as constructed above is referred to as a **BFV-complex** *associated to (E, Π).*

We remark that there are several BFV-complexes associated to (E, Π). However in [Sch] it is shown that different choices of a connection on $E \to S$ and of the BFV-charge Ω yield isomorphic differential graded Poisson algebras.

Corollary 3. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold. Assume (E, Π) is a Poisson manifold and S a coisotropic submanifold.*

The cohomology of $(BFV(E, \Pi), D_{BFV})$ is naturally isomorphic to the Lie algebroid cohomology of S introduced in Subsect. 2.5.

Proof. We use the filtration of $(BFV(E, \Pi), D_{BFV})$ given by $BFV^{(\geq q,\bullet)}(E, \Pi)$ to obtain a spectral sequence. Decomposing D_{BFV} with respect to the degree q yields $\sum_{k\geq 0} \delta_k$ with $\delta_0 = \delta = [\Omega_0, -]_G$. In the proof of Proposition 2 the isomorphism $H(BFV(E, \Pi), \delta) \cong \mathcal{C}^\infty(E^*[1])$ was established. This means that the spectral sequence under consideration collapses after one step and so $H(BFV(E, \Pi), D_{BFV})$ is naturally isomorphic to the next sheet of the spectral sequence. Hence we have to compute the cohomology of $\mathcal{C}^\infty(E^*[1])$ with respect to the induced differential to obtain $H(BFV(E, \Pi), D_{BFV})$.

It is straightforward to check that the induced differential does not depend on the particular choice of Ω and that it is given by the restriction of $\delta_1 := [\Omega_0, -]_{\iota_\nabla(\Pi)} + [\Omega_1, -]_G$ to $\mathcal{C}^\infty(E^*[1]) = \Gamma(\bigwedge E)$. A possible choice of Ω_1 is given by $-h(1/2[\Omega_0, \Omega_0]_{\iota_\nabla(\Pi)})$ with h being the homotopy defined in the proof of Proposition 2. In the local coordinates used in the proof of Proposition 2 the induced differential is given by

$$[\delta_1] = c_i \left(\Pi^{i\beta}|_S\right) \frac{\vec{\partial}}{\partial x^\beta} - \frac{1}{2}\left(\frac{\partial \Pi^{ij}}{\partial y^k}|_S\right) c_i c_j \frac{\vec{\partial}}{\partial c_k} \tag{3}$$

which coincides with the Lie algebroid differential ∂_S. Hence the second sheet of the collapsing spectral sequence associated to $(BFV(E, \Pi), D_{BFV})$ is equal to the Lie algebroid cocomplex $(\Gamma(\bigwedge E), \partial_S)$ associated to S. Consequently there is an isomorphism between $H(BFV(E, \Pi), D_{BFV})$ and the Lie algebroid cohomology of S. □

In particular one obtains

$$\begin{aligned} &H^0(BFV(E, \Pi), D_{BFV}) \cong \mathcal{C}^\infty(\underline{S}) \\ &\quad = \{f \in \mathcal{C}^\infty(S) : X(f) = 0 \text{ for all } X \in \Gamma(\mathcal{F}_S)\}. \end{aligned}$$

Due to the compatibility between D_{BFV} and the BFV-bracket $[-, -]_{BFV}$, the cohomology $H(BFV(E, \Pi), D_{BFV})$ carries the structure of a graded Poisson algebra. This structure restricts to the structure of a Poisson algebra on $H^0(BFV(E, \Pi), D_{BFV}) \cong \mathcal{C}^\infty(\underline{S})$. It is easy to show that

Lemma 7. *The algebra isomorphism $H^0(BFV(E, \Pi), D_{BFV}) \cong \mathcal{C}^\infty(\underline{S})$ maps the Poisson bracket induced from $[-, -]_{BFV}$ to $[-, -]_{\underline{S}}$ defined in Subsect. 2.5.*

Hence the BFV-complex $(BFV(E, \Pi), D_{BFV}, [-, -]_{BFV})$ can be thought of as some kind of "resolution" of the Poisson algebra $(\mathcal{C}^\infty(\underline{S}), [-, -]_{\underline{S}})$.

4. Connection to the Strong Homotopy Lie Algebroid

Let S be a coisotropic submanifold of a smooth finite dimensional Poisson manifold (M, Π). In Sect. 3 a differential graded Poisson algebra $(BFV(E, \Pi), D_{BFV}, [-, -]_{BFV})$ was constructed such that the degree zero cohomology $H^0(BFV(E, \Pi), D_{BFV})$ is isomorphic to $\mathcal{C}^\infty(\underline{S})$ as an algebra and the Poisson bracket induced from $[-, -]_{BFV}$ coincides with $[-, -]_{\underline{S}}$.

There is another "resolution" of the Poisson algebra $(\mathcal{C}^\infty(\underline{S}), [-,-]_{\underline{S}})$ given by the Lie algebroid complex associated to S, enriched with compatible higher operations. This structure was found by Oh and Park ([OP]) in the symplectic setting and called "strong homotopy Lie algebroid" there. It can also be derived as the classical limit of the Poisson Sigma model with boundary conditions given by S ([CF]). Our main aim is to show that the strong homotopy Lie algebroid is equivalent to $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$ in the appropriate sense: they are L_∞ quasi-isomorphic, see Theorem 5 in Subsect. 4.2. We remark that there is a connection between these algebraic structures and deformations of S, see [OP] and Sect. 5. Moreover Kieserman showed in [Ki] that they capture very subtle properties of the foliation $\mathcal{F}_S := \Pi^\#(N^*S)$ associated to S.

4.1. The strong homotopy Lie algebroid. We follow the presentation in [CF and Ca] where the connection to the derived brackets formalism ([V]) was made explicit.

Let $S \hookrightarrow M$ be a submanifold of a smooth finite dimensional Poisson manifold (M,Π). By choosing an embedding of the normal bundle of S as a tubular neighbourhood inside M we obtain a finite rank vector bundle $E \xrightarrow{p} S$ equipped with a Poisson bivector field. We denote the embedding of S into E as the zero section by i. Abusing notation we denote the Poisson bivector field on E by Π. We remark that there is a natural identification $E \cong NS$ (NS being the normal bundle of S in E).

There is a natural projection $pr : \mathcal{V}(E) \to \Gamma(\bigwedge E)$ given by the unique algebra morphism extending $f \mapsto f \circ i$ on $\mathcal{C}^\infty(E)$ and

$$\Gamma(TE) \to \Gamma(T_S E) \to \Gamma(E),$$

where $E \to S$ is identified with the vertical part of $T_S E \to S$. This projection admits a section $s : \Gamma(\bigwedge E) \to \mathcal{V}(E)$: on functions $g \in \mathcal{C}^\infty(S)$ it is given by $s(g) := g \circ p$ and on elements $X \in \Gamma(E)$ one defines $s(X)$ to be the unique vertical extension of X that is constant along fibres of $E \to S$.

One checks that $s(\Gamma(\bigwedge E)) \hookrightarrow \mathcal{V}(E)$ is an abelian Lie subalgebra of the graded Lie algebra $(\mathcal{V}(E)[1], [-,-]_{SN})$. Moreover $ker(pr)[1]$ is a Lie subalgebra and $\mathcal{V}(E) = ker(pr) \oplus s(\Gamma(\bigwedge E))$. Consequently

$$(\mathcal{V}(E)[1], \Gamma(\wedge E)[1], pr[1])$$

is a V-algebra (Definition 4).

The Poisson bivector field Π on E can be interpreted as a MC-element of $(\mathcal{V}(E)[1], [-,-]_{SN})$. By Theorem 1 the derived brackets associated to the Poisson bivector field

$$\hat{\mu}^k := D^k_\Pi : (\Gamma(\wedge E)[1])^{\otimes k} \to \Gamma(\wedge E)[2] \tag{4}$$

define the structure of a (possibly non-flat) $L_\infty[1]$-algebra on $\Gamma(\bigwedge E)[1]$. This corresponds to the structure of a (possibly non-flat) L_∞-algebra on $\Gamma(\bigwedge E)$. We denote the structure maps of the L_∞-algebra by $(\mu^k)_{k\in\mathbb{N}}$.

The submanifold S is coisotropic if and only if $pr(\Pi) = 0$. In this case the L_∞-algebra is flat (i.e. the zero order component $\mu^0 \in \Gamma(\bigwedge^2 E)$ vanishes) and μ^1 coincides with the Lie algebroid differential ∂_S associated to S (see Subsect. 2.5). Hence:

Theorem 3. *Let S be a coisotropic submanifold of a smooth finite dimensional Poisson manifold (M,Π). Then $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \cdots)$ constructed as above is an L_∞-algebra extending the Lie algebroid complex associated to S.*

This theorem first appeared in [OP] in the symplectic setting.

Definition 10. *Let S be a coisotropic submanifold of a smooth finite dimensional Poisson manifold (M, Π). The* **strong homotopy Lie algebroid** *associated to S is the L_∞-algebra $(\Gamma(\bigwedge NS), \partial_S = \mu^1, \mu^2, \cdots)$.*

Since $(\mathcal{V}(E)[1], [-,-]_{SN})$ is a Gerstenhaber algebra and pr and s are morphisms of algebras, the structure maps μ^k are graded multiderivations with respect to the graded algebra structure: i.e.

$$\begin{aligned}\mu^k(a_1 \otimes \cdots \otimes a_{k-1} \otimes a \cdot b) \;=\; & \mu^k(a_1 \otimes \cdots \otimes a_{k-1} \otimes a) \cdot b \\ & + (-1)^{(|a_1|+\cdots+|a_{k-1}|+2-n)|a|} a \cdot \mu^k(a_1 \otimes \cdots \otimes a_{k-1} \otimes b)\end{aligned} \tag{5}$$

holds for all k and arbitrary homogeneous elements $a_1, \ldots, a_{k-1}, a, b$ of $\Gamma(\bigwedge E)$. In [CF] L_∞-algebras on graded algebras with this property were called *P_∞-algebras*.

We remark that the derived brackets μ^k depend in general on the choice of embedding $\phi : E \hookrightarrow M$. However it was proved in [OP] in the symplectic case and in [CS] in the Poisson case and for arbitrary submanifolds (not necessary coisotropic) that different choices lead to L_∞-isomorphic L_∞-algebras:

Theorem 4. *The L_∞-algebra structures constructed on $\Gamma(\bigwedge(NS))$ with the help of two different embeddings of NS into M as tubular neighbourhoods of S are L_∞-isomorphic.*

Let S be a coisotropic submanifold of (M, Π). By Theorem 3 there is a nontrivial extension of the Lie algebroid complex $(\Gamma(\bigwedge NS), \partial_S)$ associated to S to an L_∞-algebra. As observed in Subsect. 2.5 the zero Lie algebroid cohomology $H^0(\Gamma(\bigwedge NS), \partial_S)$ is given by $\mathcal{C}^\infty(\underline{S})$. The binary operation μ^2 descends to cohomology where it induces a Lie bracket. Since μ^2 is a graded biderivation with respect to the graded algebra structure the induced Lie bracket will be a biderivation, i.e. $\mathcal{C}^\infty(\underline{S})$ inherits a Poisson bracket. A computation shows that

Lemma 8. *The algebra isomorphism $H^0(\Gamma(\bigwedge NS), \partial_S) \cong \mathcal{C}^\infty(\underline{S})$ maps the Poisson bracket induced from μ^2 to $[-,-]_{\underline{S}}$ as defined in Subsect. 2.5.*

Consequently the P_∞-algebra $(\Gamma(\bigwedge NS), \partial_S = \mu^1, \mu^2, \ldots)$ is a resolution of the Poisson algebra $(\mathcal{C}^\infty(\underline{S}), [-,-]_{\underline{S}})$.

4.2. Relation of the two structures. Let S be a coisotropic submanifold of a smooth finite dimensional Poisson manifold (M, Π). Lemma 7 in Subsect. 3.3 established that the differential graded Poisson algebra $(BFV(E, \Pi), D_{BFV}, [-,-]_{BFV})$ can be interpreted as some kind of "resolution" of the Poisson algebra $(\mathcal{C}^\infty(\underline{S}), [-,-]_{\underline{S}})$ introduced in Subsect. 2.5. The same is true for the strong homotopy Lie algebroid $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \ldots)$ constructed in Subsect. 4.1 (see Lemma 8). Moreover Corollary 3 in Subsect. 3.3 established an isomorphism of graded algebras $H^\bullet(BFV(E, \Pi), D_{BFV}) \cong H^\bullet(\Gamma(\bigwedge E), \partial_S)$.

A natural question to ask is how tight the connection between the BFV-complex $(BFV(E, \Pi), D_{BFV}, [-,-]_{BFV})$ and the P_∞-algebra $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \ldots)$ actually is. We provide an answer to this question:

Theorem 5. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is a coisotropic submanifold.*

Then there is an L_∞ quasi-isomorphism between the BFV-complex $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$ associated to S (Definition 9 in Subsect. 3.3) and the strong homotopy Lie algebroid associated to S, i.e. $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \dots)$ (Definition 10).

An immediate consequence of Theorem 5 is:

Corollary 4. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is a coisotropic submanifold.*

Then the formal deformation problems associated to the BFV-complex $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$ and to the strong homotopy Lie algebroid $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \dots)$ are equivalent.

Next we prove Theorem 5:

Proof. The strategy of the proof is as follows: the starting point is the BFV-complex $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$. As a by-product of the proof of Proposition 2 in Subsect. 3.3 we obtained an isomorphism of graded algebras $H^\bullet(BFV(E,\Pi), \delta) \cong \Gamma(\bigwedge^\bullet E)$. The $\bullet$ on the left-hand side refers to the grading with respect to the total degree. Recall that δ is $[\Omega_0, -]_G$, where $\Omega_0 \in BFV(E,\Pi)$ is given by the tautological section of the bundle $\mathcal{E} \to E$ and G denotes the Poisson bivector field on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ representing the fibre pairing between $\mathcal{E}^*[1]$ and $\mathcal{E}[-1]$.

More explicitly we considered pullbacks i^* and p^* along $i : E^* \hookrightarrow \mathcal{E}^*[1] \oplus \mathcal{E}[1]$ and $p : \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E^*[1]$ and a homotopy $h : BFV(E,\Pi) \to BFV(E,\Pi)[-1]$ such that $h \circ h = 0$, $i^* \circ h = 0$, $h \circ p^* = 0$ and $\delta \circ h + h \circ \delta = id - p^* \circ i^*$ hold. We summarize the situation in the following diagram:

$$(\mathcal{C}^\infty(E^*[1]), 0) \underset{i^*}{\overset{p^*}{\rightleftarrows}} (BFV(E,\Pi), \delta), h. \tag{6}$$

By Theorem 2 in Subsect. 2.3 these data can be used for homological transfer of an L_∞-algebra structure from $(BFV(E,\Pi), \delta)$ to $\mathcal{C}^\infty(E^*[1]) = \Gamma(\bigwedge E)$.

We will use these data to perform the homological transfer of the differential graded Lie algebra $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$ to $\Gamma(\bigwedge E)$ in terms of diagrams as described in Subsect. 2.3. It will turn out that no convergence issues arise and that the induced L_∞-algebra structure on $\Gamma(\bigwedge E)$ is a P_∞-algebra, i.e. the structure maps are graded multiderivations. Hence we have two P_∞-algebra structures on $\Gamma(\bigwedge E)$: one is induced from the BFV-complex and the second one is given by the strong homotopy Lie algebroid associated to S. Since $\Gamma(\bigwedge E)$ is generated by $\mathcal{C}^\infty(S)$ and $\Gamma(E)$ as a graded algebra it suffices to know the structure maps of the P_∞-algebra structures restricted to $\mathcal{C}^\infty(S)$ and $\Gamma(E)$ respectively in order to be able to reconstruct them completely. We will check that the restricted structure maps of the two P_∞-algebras coincide, hence so do the full P_∞-algebras.

Step 1) Homological transfer in terms of trees. We perform the homological transfer of the differential graded Lie algebra structure on $BFV(E,\Pi)$ along the diagram (6). What does the induced L_∞-algebra structure on $\Gamma(\bigwedge E)$ look like?

The BFV-differential $D_{BFV} = [\Omega, -]_{BFV}$ can be decomposed as $D_{BFV} = \delta + D_R$ satisfying $\delta \circ \delta = 0$ and $\delta \circ D_R + D_R \circ \delta + D_R \circ D_R = 0$. Recall that $BFV(E, \Pi)$ carries a bigrading given by $BFV^{(p,q)}(E, \Pi) := \Gamma(\bigwedge^p(\mathcal{E}) \otimes \bigwedge^q(\mathcal{E}^*))$. We have

$$\begin{aligned} &h : BFV^{(p,q)}(E, \Pi) \to BFV^{(p,q+1)}(E, \Pi), \\ &D_R : BFV^{(p,q)}(E, \Pi) \to \bigoplus_{(p'>p, q'\geq q, p'-q'=p-q)} BFV^{(p',q')}(E, \Pi) \text{ and} \\ &[-,-]_{BFV} = \left([-.-]_G + [-,-]_{\iota_\nabla(\Pi)}\right) \bmod BFV_{\geq 1}(E, \Pi). \end{aligned}$$

Following Subsect. 2.3 the induced structure maps are given in terms of oriented trees with edges decorated by non-negative integers. The set of exterior vertices decomposes into the set of leaves (with edges pointing away from them) and a unique root (with an edge pointing towards it). To each such decorated tree T a map

$$m_T := \Gamma(\bigwedge E)^{\otimes \#(leaves)} \to \Gamma(\bigwedge E)$$

is associated by the following rule: put $[-,-]_{BFV}$ at the trivalent vertices and k copies of D_R at edges decorated by the number k. Between consecutive operations $[-,-]_{BFV}$ or D_R place a homotopy $-h$. We define $\tilde{m}_T : BFV(E, \Pi)^{\otimes \#(leaves)} \to BFV(E, \Pi)$ to be the composition of all these maps in the order given by the orientation of the tree T. Then we set $m_T := i^* \circ \tilde{m}_T \circ (p^*)^{\otimes \#(leaves)}$.

Because $p^*(\Gamma(\bigwedge E)) \subset BFV^{(\bullet,0)}(E, \Pi)$ and $(i^*)^{-1}(\Gamma(\bigwedge E)) \subset BFV^{(\bullet,0)}(E, \Pi)$ the operation m_T associated to a decorated tree T can only be non-zero if the corresponding $\tilde{m}_T$ maps the subspace $(BFV^{(\bullet,0)}(E, \Pi))^{\otimes \#(leaves)}$ to a subspace having nonvanishing intersection with $BFV^{(\bullet,0)}(E, \Pi)$.

Since the homotopy h increases the ghost-momentum degree by 1 and $[-,-]_G$ is the only operation that decreases it by 1, there must be at least as many trivalent vertices decorated by $[-,-]_G$ as there are hs. From

$$\#([-,-]_G) \geq \#(h) = \#(D_R) + \#(\text{trivalent vertices}) - 1$$

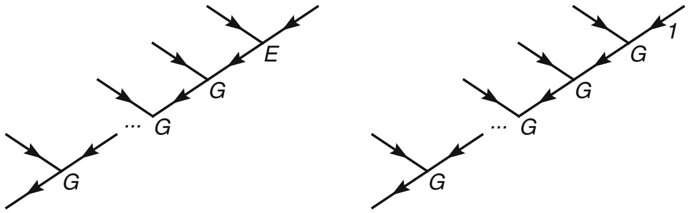
it follows that

$$\#(D_R) + \#(\text{trivalent vertices decorated by } ([-,-]_{BFV}) - ([-,-]_G)) \leq 1.$$

One can easily exclude the sharp inequality so there are two remaining cases: Either 1) all the edges of the tree are decorated by zeros. In this case exactly one of the trivalent vertices is decorated by $([-,-]_{BFV}) - ([-,-]_G)$ and the other trivalent vertices are decorated by $[-,-]_G$. Or 2) exactly one edge is decorated by 1 and all the others by zero. In this case all of the trivalent vertices are decorated by $[-,-]_G$.

Observe that in both Case 1) and 2) the part of the "exceptional" operation D_R and $([-,-]_{BFV} - [-,-]_G)$ respectively that actually contributes to m_T is the part of ghost-momentum degree 0. We decompose D_{BFV} with respect to the ghost degree $D_{BFV} = \sum_{k\geq 0} \delta_k$ with $\delta_0 = \delta$, and hence $D_R = \sum_{k\geq 1} \delta_k$. The fact that D_R is of total degree 1 implies that its component of ghost-momentum degree 0 is given by δ_1. The ghost-momentum degree 0 component of $([-,-]_{BFV}) - ([-,-]_G)$ is $[-,-]_{\iota_\nabla(\Pi)}$.

Moreover the identity $[p^*(-), p^*(-)]_G = 0$ holds because $p^*(\Gamma(\bigwedge E)) \subset BFV^{(\bullet,0)}$ (E, Π) and because $[-,-]_G$ is the graded Poisson bracket induced from the fibre pairing between $\mathcal{E}^*[1]$ and $\mathcal{E}[-1]$. Hence the only two types of trees that contribute to the induced L_∞-algebra structure on $\Gamma(\bigwedge E)$ are the following:



Here the decoration E refers to $[-,-]_{\iota_\nabla(\Pi)}$, G refers to $[-,-]_G$ and the decoration of the edges was left out whenever it is zero. We denote the maps from $(\Gamma(\bigwedge E))^{\otimes n}$ to $\Gamma(\bigwedge E)$ associated to the trees on the left-/right-hand side with n leaves by L_n and R_n respectively. Up to skew-symmetrization and sign issues these two families of maps define the induced L_∞-algebra structure on $\Gamma(\bigwedge E)$.
Step 2) P_∞-property. The L_∞-algebra structure $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \dots)$ satisfies the P_∞ property (5) as remarked before. Furthermore

Lemma 9. *The L_∞-algebra structure on $\Gamma(\bigwedge E)$ induced from the differential graded Poisson algebra $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$ satisfies the P_∞ property (5).*

Proof. We first prove that the result of the evaluation of L_n (R_n) on elements of the form

$$a_1 \otimes \cdots \otimes a_{k-1} \otimes a \cdot b \otimes a_k \cdots \otimes a_{n-1} \in \Gamma(\bigwedge E)^{\otimes n}$$

can be expressed using L_n (R_n) evaluated on $a_1 \otimes \cdots a \cdots \otimes a_{n-1}$ and on $a_1 \otimes \cdots \otimes b \otimes a_{n-1}$ only. Without loss of generality one may assume that $a_1, \dots, a_{(n-1)}, a, b$ are all homogeneous.

Consider the map L_n first and assume that $k < (n-1)$. By the graded Leibniz identity for $[-,-]_G$ we have

$$\begin{aligned} [p^*(a \cdot b), \bullet]_G &= [p^*(a) \cdot p^*(b), \bullet]_G \\ &= p^*(a) \cdot [p^*(b), \bullet]_G + (-1)^{|a||b|} p^*(b) \cdot [p^*(a), \bullet]_G. \end{aligned}$$

Recall the definition of the homotopy h given during the proof of Proposition 2 in Subsect. 3.3:

$$\begin{aligned} &h(f_{\mu_1\dots\mu_k}(x,y,c) b^{\mu_1} \cdots b^{\mu_k}) \\ &:= \sum_{1 \le \mu \le s} b^\mu \left(\int_0^1 \frac{\partial f_{\mu_1\dots\mu_k}}{\partial y^\mu}(x, t \cdot y, c) t^k dt \right) b^{\mu_1} \cdots b^{\mu_k}. \end{aligned}$$

Hence $h(p^*(X) \cdot Y) = (-1)^{|X|} p^*(X) \cdot h(Y)$ because p^*X does not depend on the coordinates y^μ and b^μ. So

$$\begin{aligned} &h([p^*(a \cdot b), \bullet]_G) \\ &= (-1)^{|a|} p^*(a) \cdot h([p^*(b), \bullet]_G) + (-1)^{(|a|+1)|b|} p^*(b) \cdot h([p^*(a), \bullet]_G) \end{aligned} \tag{7}$$

holds. Applying consecutively

(1) $[p^*(-), -]_G$ and using the graded Leibniz identity together with $[p^*(-), p^*(-)]_G = 0$; and

(2) h and using $h(X \cdot p^*(Y)) = h(X) \cdot p^*(Y)$

leads to

$$
\begin{aligned}
&L_n(a_1 \otimes \cdots \otimes a_{k-1} \otimes a \cdot b \otimes a_k \cdots \otimes a_{n-1}) \\
&\quad = (-1)^{(|a_1|+\cdots+|a_{k-1}|+k)|a|} a \cdot L_n(a_1 \otimes \cdots \otimes a_{k-1} \otimes b \otimes a_k \cdots \otimes a_{n-1}) \\
&\qquad + (-1)^{(|a_1|+\cdots+|a_{k-1}|+|a|+k)|b|} b \cdot L_n(a_1 \otimes \cdots \otimes a_{k-1} \otimes a \otimes a_k \cdots \otimes a_{n-1})
\end{aligned}
$$

for $k < (n-1)$. By similar reasoning this formula can be extended to the cases $k = (n-1)$ and $k = n$.

We claim that

$$
\begin{aligned}
&R_n(a_1 \otimes \cdots \otimes a_{k-1} \otimes a \cdot b \otimes a_k \cdots \otimes a_{n-1}) \\
&\quad = (-1)^{(|a_1|+\cdots+|a_{k-1}|+k)|a|} a \cdot R_n(a_1 \otimes \cdots \otimes a_{k-1} \otimes b \otimes a_k \cdots \otimes a_{n-1}) \\
&\qquad + (-1)^{(|a_1|+\cdots+|a_{k-1}|+|a|+k)|b|} b \cdot R_n(a_1 \otimes \cdots \otimes a_{k-1} \otimes a \otimes a_k \cdots \otimes a_{n-1})
\end{aligned}
$$

holds as well. For $k < n$ the arguments previously applied to L_n go through. For the case $k = n$ we make use of the explicit formula for δ_1 which was derived in the proof of Corollary 3 in Subsect. 3.3:

$$\delta_1 = [\Omega_0, -]_{\iota_\nabla(\Pi)} + [\Omega_1, -]_G.$$

Hence $\delta_1(p^*(a \cdot b)) = \delta_1(p^*(a)) \cdot p^*(b) + (-1)^{|a|} p^*(a) \cdot \delta_1(p^*(b))$ and applying the established computation rules for h and $[p^*(-), -]_G$ yields the claimed formula for R_n.

If one takes the signs arising from the décalage-isomorphism and graded symmetrization into account one obtains the signs as stated in (5). $\square$

Step 3) Localization. The graded commutative associative algebra $\mathcal{C}^\infty(E^*[1]) = \Gamma(\bigwedge E)$ is generated by elements of degree 0 and 1, i.e. by $\mathcal{C}^\infty(S)$ and $\Gamma(E)$. Hence it is enough to know L_n and R_n restricted to $(\mathcal{C}^\infty(S) \oplus \Gamma(E))^{\otimes n} \subset (\Gamma(\bigwedge E))^{\otimes n}$ by Lemma 9. Since $\Gamma(\bigwedge E)$ is concentrated in non-negative degrees and the total degree of L_n and R_n is $(2-n)$, it suffices to know L_n and R_n on elements of one of the following types:

A') $\gamma^1 \otimes \cdots \otimes \gamma^n$ for $\gamma^i \in \Gamma(E)$,
B') $\gamma^1 \otimes \cdots \otimes \gamma^{(k-1)} \otimes f \otimes \gamma^k \otimes \cdots \otimes \gamma^{(n-1)}$ for $\gamma^i \in \Gamma(E)$, $f \in \mathcal{C}^\infty(S)$,
C') $\gamma^1 \otimes \cdots \otimes \gamma^{(k-1)} \otimes f \otimes \gamma^k \otimes \cdots \otimes \gamma^{(k+l-1)} \otimes g \otimes \gamma^{(k+l)} \otimes \cdots \otimes \gamma^{(n-2)}$ for $\gamma^i \in \Gamma(E)$, $f, g \in \mathcal{C}^\infty(S)$.

We choose a trivializing cover $\mathcal{U} := (U_\alpha)_{\alpha \in A}$ for the vector bundle $E \to S$. Let $(\rho_\alpha)_{\alpha \in A}$ be a partition of unity subordinated to $\mathcal{U}$, i.e. *a)* $\rho_\alpha \in \mathcal{C}^\infty(S)$, *b)*$supp(\rho_\alpha) \subset U_\alpha$ for every $\alpha \in A$, *c)* $(\rho_\alpha)_{\alpha \in A}$ is locally finite (for every $x \in S$ there is an open neighbourhood U such that there are only finitely many $\alpha \in A$ with $\rho_\alpha|_U \neq 0$) and *d)* $\sum_{\alpha \in A} \rho_\alpha = 1$.

For an arbitrary $f \in \mathcal{C}^\infty S$ we write

$$f = \left(\sum_{\alpha \in A} \rho_\alpha \right) f = \sum_{\alpha \in A} (\rho_\alpha f) =: \sum_{\alpha \in A} f_\alpha,$$

where f_α is supported on U_α. Similarly we get $\gamma = \sum_{\alpha \in A} \gamma_\alpha$ for any section $\gamma \in \Gamma(E)$. Since $\mathcal{U}$ is a collection of trivializing neighbourhoods of the vector bundle E we can choose a local frame $(e_1^\alpha, \dots, e_s^\alpha)$ of E restricted to U_α. The section γ_α is supported on U_α and hence there are local functions $(w_\alpha^1, \dots, w_\alpha^r)$ such that

$$\gamma_\alpha = \sum_{j=1}^{s} w_\alpha^j e_j^\alpha.$$

Using this decomposition of smooth functions and sections of E on elements of $(\mathcal{C}^\infty(S) \oplus \Gamma(E))^{\otimes n}$ of type A'), B') or C') shows that L_n and R_n are totally determined by evaluating them for arbitrary $\alpha \in A$ on elements of the form

A) $e_{j_1}^\alpha \otimes \cdots \otimes e_{j_n}^\alpha$,
B) $e_{j_1}^\alpha \otimes \cdots \otimes e_{j_{(k-1)}}^\alpha \otimes f \otimes e_{j_k}^\alpha \otimes \cdots \otimes e_{j_{(n-1)}}^\alpha$ with $f \in \mathcal{C}^\infty(U_\alpha)$,
C) $e_{j_1}^\alpha \otimes \cdots \otimes e_{j_{(k-1)}}^\alpha \otimes f \otimes e_{j_k}^\alpha \otimes \cdots \otimes e_{j_{(k+l-1)}}^\alpha \otimes g \otimes e_{j_{(k+l)}}^\alpha \otimes \cdots \otimes e_{j_{(n-2)}}^\alpha$ with $f, g \in \mathcal{C}^\infty(U_\alpha)$.

Since we only use the P_∞ property and the total degrees of the structure maps L_n and R_n, the same is true for the structure maps μ^n of the strong homotopy Lie algebroid $(\Gamma(\bigwedge E), \partial_S = \mu^1, \mu^2, \dots)$ associated to S.
Step 4) Comparison of the restricted structure maps. Let U_α be an open subset of the trivializing cover $\mathcal{U}$. The aim is to compute explicit coordinate expressions on U_α for the restricted structure maps of the strong homotopy Lie algebroid and the structure induced from the BFV-complex respectively.

Let $(x^\beta)_{\beta=1,\dots s}$ be coordinates for S and $(y^j)_{j=1,\dots e}$ linear fibre coordinates along E. We have to consider the graded Lie algebra $\mathcal{V}(E|_{U_\alpha})[1]$ with the bracket given by

$$\left[\frac{\partial}{\partial x^\alpha}, x^\beta\right]_{SN} = \delta_\alpha^\beta, \quad \left[\frac{\partial}{\partial y^i}, y^j\right]_{SN} = \delta_i^j.$$

The Poisson bivector field Π is given by

$$\frac{1}{2}\Pi^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + \Pi^{\alpha j}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^j} + \frac{1}{2}\Pi^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}.$$

A straightforward computation of the restricted structure maps $\hat{\mu}^k$ of the $L_\infty[1]$-algebra structure on $\Gamma(\bigwedge E|_{U_\alpha})[1]$ yields

$$\hat{\mu}^k\left(\frac{\partial}{\partial y^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{j_k}}\right) = (-1)^k \frac{1}{2}\left(\frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_k}}\left(\Pi^{il}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^l}\right)\right)|_S,$$
$$\hat{\mu}^k\left(\frac{\partial}{\partial y^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{j_{(k-1)}}} \otimes f(x)\right) = (-1)^k\left(\frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_{(k-1)}}}\left(\Pi^{\alpha l}\frac{\partial f(x)}{\partial x^\alpha}\frac{\partial}{\partial y^l}\right)\right)|_S,$$
$$\hat{\mu}^k\left(\frac{\partial}{\partial y^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{j_{(k-2)}}} \otimes f(x) \otimes g(x)\right)$$
$$= (-1)^{(k-1)}\left(\frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_{(k-2)}}}\left(\Pi^{\alpha\beta}\frac{\partial f(x)}{\partial x^\alpha}\frac{\partial g(x)}{\partial x^\beta}\right)\right)|_S.$$

Only the last expression picks up a sign under the décalage-isomorphism: the exponent changes from $(k-1)$ to k.

To obtain concrete formulae for the induced L_∞-algebra structure we first make some general observations on the induced structure maps. All the operations D_R, h, $[-,-]_G$

and $[-,-]_{\iota_\nabla(\Pi)}$ are (multi-)differential operators and the surjection from $BFV(E,\Pi)$ to its cohomology $\Gamma(\bigwedge E)$ involves the evaluation of sections at $S \hookrightarrow E$. It follows that the induced structure maps only depend on the jet-expansion of Π in transversal directions and that the homotopy h can be replaced by its jet-version. For convenience let us introduce the following local coordinates: $(x^\beta)_{\beta=1,\dots,s}$ on S, linear fibre coordinates $(y^j)_{j=1,\dots,e}$ along E, $(c_j)_{j=1,\dots,e}$ along $\mathcal{E}^*[1]$ and $(b^j)_{j=1,\dots,e}$ along $\mathcal{E}[-1]$. In these local coordinates the jet-version of the homotopy reads

$$\begin{aligned}&\hat{h}(f_{j_1\dots j_k}(x,y,c)b^{j_1}\cdots b^{j_k})\\ &:= \sum_{1\leq\mu\leq e}\frac{1}{N(f)+k}b^\mu\left(\frac{\partial f_{j_1\dots j_k}}{\partial y^\mu}(x,y,c)\right)b^{j_1}\cdots b^{j_k},\end{aligned}$$

where $N(f)$ is the polynomial degree of f with respect to the transverse directions $(y^j)_{j=1,\dots e}$.

In local coordinates the horizontal lift $\iota_\nabla(\Pi)$ of Π is given by

$$\begin{aligned}&\frac{1}{2}\Pi^{\alpha\beta}\left(\frac{\partial}{\partial x^\alpha}+\Gamma^s_{\alpha r}c_s\frac{\partial}{\partial c_r}-\Gamma^s_{\alpha r}b^r\frac{\partial}{\partial b^s}\right)\left(\frac{\partial}{\partial x^\beta}+\Gamma^n_{\beta m}c_n\frac{\partial}{\partial c_m}-\Gamma^n_{\beta m}b^m\frac{\partial}{\partial b^n}\right)\\ &+\Pi^{\alpha j}\left(\frac{\partial}{\partial x^\alpha}+\Gamma^s_{\alpha r}c_s\frac{\partial}{\partial c_r}-\Gamma^s_{\alpha r}b^r\frac{\partial}{\partial b^s}\right)\frac{\partial}{\partial y^j}+\frac{1}{2}\Pi^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}.\end{aligned}$$

Here Γ denotes the Christoffel symbols of the pull back connection on $\mathcal{E}[1]\oplus\mathcal{E}^*[-1]$. Moreover the restriction of

$$\delta_1(-) = [\Omega_0,-]_{\iota_\nabla(\Pi)} + [\Omega_1,-]_G$$

(with $\Omega_1 := -\frac{1}{2}h([\Omega_0,\Omega_0]_{\iota_\nabla(\Pi)})$) to $\Gamma(E) \hookrightarrow BFV(E,\Pi)$ reads

$$\begin{aligned}&\Gamma^s_{\alpha r}y^r c_s\Pi^{\alpha\beta}\left(\frac{\partial}{\partial x^\beta}+\Gamma^n_{\beta m}c_n\frac{\partial}{\partial c_m}\right)+c_m\Pi^{m\alpha}\left(\frac{\partial}{\partial x^\alpha}+\Gamma^s_{\alpha r}c_s\frac{\partial}{\partial c_r}\right)\\ &-\frac{\partial}{\partial b^\mu}\left(\hat{h}\left(\frac{1}{2}\Pi^{\alpha\beta}\Gamma^s_{\alpha r}y^r c_s\Gamma^n_{\beta m}y^m c_n+\Pi^{\alpha k}\Gamma^s_{\beta r}y^r c_s c_k+\frac{1}{2}\Pi^{ij}c_ic_j\right)\right)\frac{\partial}{\partial c_\mu}.\end{aligned}$$

A straightforward but lengthy calculation with the restricted structure maps of the induced P_∞-algebra structure shows that all contributions involving Christoffel-symbols cancel each other and that the formulae reduce to the local expressions for μ^k derived above. □

5. The Deformation Problem

A relation between BFV-complexes (see Definition 9 in Subsect. 3.3) and so-called coisotropic graphs is presented. More precisely Theorem 6 in Subsect. 5.2 establishes a one-to-one correspondence between equivalence classes of normalized MC-elements of a BFV-complex and coisotropic graphs. Although the BFV-complex is L_∞ quasi-isomorphic to the strong homotopy Lie algebroid according to Theorem 5 in Subsect. 4.2 the two structures capture different information in the non-formal regime. As a demonstration of this phenomenon we provide a simple example of a coisotropic submanifold inside a Poisson manifold where the strong homotopy Lie algebroid fails to detect obstructions to coisotropic deformations. In the formal setting the normalization condition on MC-elements introduced in Subsect. 5.2 turns out to be superfluous. Furthermore we use the BFV-complex to treat an example which was also considered in [OP and Z] and recover some of the results derived there.

5.1. Deformations of coisotropic submanifolds. Let S be a coisotropic submanifold of a smooth finite dimensional Poisson manifold (M, Π). We fix an embedding of the normal bundle of S into M. Hence we obtain a vector bundle $E \to S$ such that E is equipped with a Poisson bivector field Π for which $S \to E$ is coisotropic.

Consider all embedded submanifolds of E. These form a subset $\mathcal{S}(E)$ of the set $\mathcal{P}(E)$ of all subsets of E. There is a map

$$\begin{aligned} \widetilde{graph} : \Gamma(E) &\to \mathcal{S}(E) \\ \mu &\mapsto S_\mu := \{(x, -\mu(x)) \in E : x \in S\}. \end{aligned}$$

We denote the intersection of the image of $\widetilde{graph}$ with the space of all coisotropic submanifolds of (E, Π) by $\mathcal{C}(E, \Pi)$, the set of *coisotropic graphs*.

Given the set $\mathcal{C}(E, \Pi)$ one can ask the question whether it is representable in an algebraic way. The precise meaning of this is the following: consider a differential graded Lie algebra $(V, d, [-, -])$. In Subsect. 2.1 the set of MC-elements of $(V, d, [-, -])$ was defined to be

$$MC(V) := \{\beta \in V_1 : d(\beta) + \frac{1}{2}[\beta, \beta] = 0\}.$$

One can ask whether there is a differential graded Lie algebra (more generally an L_∞-algebra) V such that $MC(V) = \mathcal{C}(E, \Pi)$. We will show in Subsect. 5.2 that this is the case if one chooses the differential graded Poisson algebra $(BFV(E, \Pi), D_{BFV}, [-, -]_{BFV})$ and imposes a normalization condition.

We remark that a very special case of this situation occurs when one considers Lagrangian submanifolds of symplectic manifolds. Let (M, Π) be symplectic, i.e. $\Pi^\#$ is assumed to be an isomorphism of bundles. Consequently $dim(M)$ must be $2n$ for some $n \in \mathbb{N}$. A coisotropic submanifold L of M is called *Lagrangian* if $dim(L) = n$. Using an extension of Darboux's Theorem due to Weinstein ([W1]) one can show that there is an embedding of the normal bundle E of L into M as a tubular neighbourhood such that

$$\mathcal{C}(E, \Pi) \cong \{\gamma \in \Omega^1(L) : d_{DR}(\gamma) = 0\}.$$

A generalization of this statement to coisotropic submanifolds S of symplectic manifolds (M, Π) was investigated in [OP]. It was shown that

$$\mathcal{C}^c(E, \Pi) \cong MC^c(\Gamma(\wedge E)),$$

where $\Gamma(\bigwedge E)$ is equipped with the structure of the strong homotopy Lie algebroid associated to S, see Definition 10 in Subsect. 4.1. The superscript c stands for "close" and refers to the fact that only sections sufficiently close to the zero section are taken into account.

The arguments in [OP] heavily rely on Gotay's study of coisotropic submanifolds inside symplectic manifolds, see [G]. Gotay showed that the pull back of the symplectic form to the submanifold determines the symplectic form on a tubular neighbourhood (up to neighbourhood equivalence). In particular this implies that there is an embedding of the normal bundle of a coisotropic submanifold into the symplectic manifold such that the Poisson bivector field is polynomial in fibre directions. This fails in the Poisson case.

The following example shows that the results concerning the deformation problem of coisotropic submanifolds inside symplectic manifolds mentioned above do not carry over to the Poisson case: Consider $\mathbb{R}^2$ equipped with the smooth Poisson bivector field

$$\Pi := \begin{cases} 0 & \text{for} \quad (x, y) = (0, 0) \\ exp\big(-\frac{1}{x^2+y^2}\big)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} & \text{for} \quad (x, y) \neq (0, 0). \end{cases}$$

It vanishes to all orders at $(0, 0)$ but is symplectic on $\mathbb{R}^2 \backslash \{(0, 0)\}$. The point $(0, 0)$ is a coisotropic submanifold and obviously

$$\mathcal{C}(\mathbb{R}^2, \Pi) = \{(0, 0)\}.$$

However the strong homotopy Lie algebroid associated to $(0, 0)$ is $(\mathbb{R}^2, 0, \dots)$, so

$$MC(\mathbb{R}^2) \cong \mathbb{R}^2.$$

Hence $\mathcal{C}(\mathbb{R}^2, \Pi)$ is not isomorphic to $MC(\mathbb{R}^2)$.

5.2. (Normalized) MC-elements and the gauge action. Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is a coisotropic submanifold. The aim is to study the set of MC-elements and the deformation problem associated to the BFV-complex $(BFV(E, \Pi), D_{BFV}, [-, -]_{BFV})$, see Definition 9 in Subsect. 3.3.

Recall that the BFV-differential D_{BFV} is given by the adjoint action of a special degree one element Ω which was constructed in Subsect. 3.3. Consequently the MC-equation for the BFV-complex can be written as

$$[\Omega + \beta, \Omega + \beta]_{BFV} = 0 \tag{8}$$

for $\beta \in BFV^1(E, \Pi)$.

Definition 11. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is a coisotropic submanifold.*

The set of **algebraic Maurer-Cartan elements** *associated to the BFV-complex $(BFV(E, \Pi), D_{BFV}(-) = [\Omega, -]_{BFV}, [-, -]_{BFV})$ is given by*

$$\mathcal{D}_{alg}(E, \Pi) := \{\beta \in BFV^1(E, \Pi) : [\Omega + \beta, \Omega + \beta]_{BFV} = 0\}.$$

We remark that $\mathcal{D}_{alg}(E, \Pi)$ contains elements that do not possess a clear geometric interpretation. Moreover $-\Omega$ is an element of $\mathcal{D}_{alg}(E, \Pi)$ that corresponds to the fact that E is a coisotropic submanifold of (E, Π). However we would prefer to study coisotropic submanifolds of E that are "similar" to S only, so they should at least be of the same dimension as S.

These defects can be cured with the help of a normalization condition on β. By definition

$$BFV^1(E, \Pi) := \Gamma\left(\bigoplus_{k \geq 0} (\wedge^{(k+1)}\mathcal{E} \otimes \wedge^k \mathcal{E}^*)\right),$$

where $\mathcal{E} \to E$ is the pullback bundle of $E \to S$ by $E \to S$. Hence $\beta \in BFV^1(E, \Pi)$ decomposes uniquely into

$$\beta = \sum_{k \geq 0} \beta_k$$

with $\beta_k \in \Gamma(\bigwedge^{(k+1)} \mathcal{E} \otimes \bigwedge^k \mathcal{E}^*) =: BFV^{k+1,k}(E, \Pi)$. In particular we obtain a map

$$T : BFV^1(E, \Pi) \to \Gamma(\mathcal{E})$$
$$\beta \mapsto \beta_0$$

which we call the *truncation map*. Furthermore there is a natural map $p^! : \Gamma(E) \to \Gamma(\mathcal{E})$ given by the pull back of sections.

Definition 12. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is a coisotropic submanifold.*

The set of **normalized Maurer-Cartan elements** *associated to the BFV-complex $(BFV(E, \Pi), D_{BFV}(-) = [\Omega, -]_{BFV}, [-, -]_{BFV})$ is given by*

$$\mathcal{D}_{nor}(E, \Pi) := \mathcal{D}_{alg}(E, \Pi) \cap T^{-1}(p^!(\Gamma(E)).$$

Assume that $\beta \in \mathcal{D}_{nor}(E, \Pi)$, consequently

$$T(\Omega + \beta) = \Omega_0 + p^!(\mu)$$

for a unique $\mu \in \Gamma(E)$. It is straightforward to check that the set $zero(\Omega_0 + p^!(\mu))$ of zeros of the section $\Omega_0 + p^!(\mu)$ is given by the submanifold $\widetilde{graph}(\mu) =: S_\mu$ of E. In conclusion we obtain a map

$$\mathcal{Z} : \mathcal{D}_{nor}(E, \Pi) \to \mathcal{S}(E)$$
$$\beta \mapsto zero(T(\Omega + \beta))$$

with $\mathcal{S}(E)$ denoting the set of embedded submanifolds of E.

We consider the adjoint action of $BFV^0(E, \Pi)$ on $BFV(E, \Pi)$. The Poisson algebra $BFV^0(E, \Pi)$ comes equipped with a filtration by Poisson subalgebras $BFV^0_{\geq r}(E, \Pi) := BFV^0(E, \Pi) \cap BFV_{\geq r}(E, \Pi)$, where $BFV_{\geq r}(E, \Pi)$ was defined as $\Gamma(\bigwedge \mathcal{E} \otimes \bigwedge^{\geq r} \mathcal{E}^*)$. Let $\widetilde{BFV}(E, \Pi)$ be the space of smooth sections of the pull back bundle of $\bigwedge \mathcal{E} \otimes \bigwedge \mathcal{E}^*$ under $E \times [0, 1] \to E$. This graded algebra inherits the structure of a graded Poisson algebra and all the gradings (by ghost degree, ghost-momentum degree, total degree) and the filtration by $BFV_{\geq r}(E, \Pi)$ from $BFV(E, \Pi)$. In particular we obtain a Poisson algebra $\widetilde{BFV}^0(E, \Pi)$ which is filtered by Poisson subalgebras $\widetilde{BFV}^0_{\geq r}(E, \Pi)$. It acts on $BFV(E, \Pi)$ by time-dependent endomorphisms which are derivations for both the associative algebra structure and the graded Poisson bracket $[-, -]_{BFV}$. We denote the Lie algebra of such time-dependent endomorphisms given by elements of $\widetilde{BFV}^0(E, \Pi)$ by $\underline{\mathrm{inn}}(BFV(E, \Pi))$. Such endomorphisms can be interpreted as time-dependent vector fields on the smooth graded manifold $\mathcal{E}[1] \oplus \mathcal{E}^*[-1]$ that preserve the Poisson bivector field $\hat{\Pi}$, see Corollary 1 in Subsect. 3.2.

The group of *automorphisms* $Aut(BFV(E, \Pi))$ is the space of all isomorphisms of the unital graded commutative associative algebra $BFV(E, \Pi)$ that preserve the total degree and the graded Poisson bracket $[-, -]_{BFV}$. An automorphism ψ is called *inner* if it is generated by an element of $\underline{\mathrm{inn}}(BFV(E, \Pi))$. More precisely we impose that

- there is a family of automorphisms $(\psi_t)_{t\in[0,1]}$ with $\psi_0 = id$ and $\psi_1 = \psi$,
- there is a morphism of unital graded commutative associative algebras and Poisson algebras $\hat{\psi} : BFV(E,\Pi) \to \widetilde{BFV}(E,\Pi)$

such that

- the composition of $\hat{\psi}$ with the pull back along the inclusion $E \times \{t\} \to E \times [0,1]$ coincides with ψ_t,
- the time-dependent derivation of $BFV(E,\Pi)$ that maps β to

$$(e,s) \mapsto \frac{d}{dt}|_{t=s}\left(\hat{\psi}(\beta)|_e\right)$$

  is an element of $\underline{\mathrm{inn}}(BFV(E,\Pi))$.

We denote the subset of inner automorphisms of $BFV(E,\Pi)$ by $Inn(BFV(E,\Pi))$ which can be checked to be a subgroup of $Aut(BFV(E,\Pi))$. Moreover the filtration of $\widetilde{BFV}^0(E,\Pi)$ by the Poisson subalgebras $\widetilde{BFV}^0_{\geq r}(E,\Pi)$ yields a filtration of $Inn(BFV(E,\Pi))$ by subgroups which we denote by $Inn^{\geq r}(BFV(E,\Pi))$.

The group $Aut(BFV(E,\Pi))$ acts on $\mathcal{D}_{alg}(E,\Pi)$ via

$$(\hat{\Theta},\alpha) \mapsto \hat{\Theta}(\Omega+\alpha) - \Omega$$

and consequently so do all the groups $Inn^{\geq r}(BFV(E,\Pi))$. Observe that the action of $Inn^{\geq 2}(BFV(E,\Pi))$ on $\mathcal{D}_{alg}(E,\Pi)$ restricts to an action on $\mathcal{D}_{nor}(E,\Pi)$.

Theorem 6. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E,Π) is a Poisson manifold such that S is a coisotropic submanifold.*

Mapping elements of $BFV^1(E,\Pi)$ to the zero set of their truncation induces a bijection between

(1) $\mathcal{D}_{nor}(E,\Pi)/Inn^{\geq 2}(BFV(E,\Pi))$ *and*
(2) $\mathcal{C}(E,\Pi)$.

Proof. Claim A. An element $p^!(\mu) \in \Gamma(\mathcal{E})$ can be extended to a MC-element of $(BFV(E,\Pi), D_{BFV}, [-,-]_{BFV})$ if and only if

$$S_\mu := \{(x,-\mu(x)) \in E | x \in S\}$$

is a coisotropic submanifold of (E,Π).

Given an arbitrary $\mu \in \Gamma(E)$ we want to construct a $\beta \in \mathcal{D}_{alg}(E,\Pi)$ decomposing as

$$\beta = \sum_{k\geq 0} \beta_k,$$

where $\beta_k \in \Gamma(\bigwedge^{(k+1)}\mathcal{E} \otimes \bigwedge^k \mathcal{E}^*)$ such that $\beta_0 = p^!(\mu)$ holds. This is a generalization of the construction of Ω given in the proof of Proposition 2 in Subsect. 3.3.

First consider $\Omega_0 + p^!(\mu) \in \Gamma(\mathcal{E})$. Obviously

$$[\Omega_0 + p^!(\mu), \Omega_0 + p^!(\mu)]_G = 0$$

holds. Recall that G is the Poisson bivector field on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ that corresponds to the fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$. Consequently we obtain a differential

$$\delta[\mu](-) := [\Omega_0 + p^!(\mu), -]_G =: \delta(-) + \partial_\mu(-).$$

In the proof of Proposition 2 in Subsect. 3.3 a homotopy h for δ was defined satisfying $h \circ \delta + \delta \circ h = id - p^* \circ i^*$, where $p^* : \Gamma(\bigwedge E) \to BFV(E, \Pi)$ is essentially given by the pull back $p^!$ and $i^* : BFV(E, \Pi) \to \Gamma(\bigwedge E)$ is given by natural restriction and projection maps. The concrete formula of h implies that

$$h \circ \partial_\mu + \partial_\mu \circ h = 0,$$

since $p^!(\mu)$ is a section of $\Gamma(\mathcal{E}) \subset BFV(E, \Pi)$ that is constant along the fibres of $E \to S$ and consequently

$$h \circ \delta[\mu] + \delta[\mu] \circ h = id - p^* \circ i^*.$$

Observe that the maps p^* and i^* are no longer morphisms of complexes with respect to δ_μ.

Consider the diffeomorphism $q_\mu : S_\mu := \{(x, -\mu(x)) | x \in S\} \to S$ and the pull back vector bundle $q_\mu^!(E) \to S_\mu$.

Claim A.1. $H^\bullet(BFV(E, \Pi), \delta[\mu]) \cong \Gamma(\bigwedge^\bullet(q_\mu^! E))$. Since S_μ and S are diffeomorphic there is a vector bundle isomorphism between $\bigwedge q_\mu^!(E)$ and $\bigwedge E$ which induces an isomorphism ϑ of graded algebras between $\Gamma(\bigwedge q_\mu^!(E))$ and $\Gamma(\bigwedge E)$. It is straightforward to check that

$$p_\mu^* : \Gamma(\wedge q_\mu^!(E)) \xrightarrow{\vartheta} \Gamma(\wedge E) \xrightarrow{p^*} BFV(E, \Pi)$$

and

$$i_\mu^* : BFV(E, \Pi) \xrightarrow{i^*} \Gamma(\wedge E) \xrightarrow{\vartheta^{-1}} \Gamma(\wedge q_\mu^!(E))$$

are chain maps between $(BFV(E, \Pi), \delta_\mu)$ and $(\Gamma(\bigwedge p_\mu^! E), 0)$. In fact, p_μ^* is given by the unique extension of a section of $\bigwedge q_\mu^!(E)$ to a section of $\bigwedge \mathcal{E} \otimes \bigwedge \mathcal{E}^*$ that is constant along the fibres of $\mathcal{E} \to E$. Furthermore i_μ^* is given by the composition of $BFV(E, \Pi) \to \Gamma(\bigwedge \mathcal{E})$ with the evaluation at S_μ. Obviously $i_\mu^* \circ p_\mu^* = id$ and

$$h \circ \delta[\mu] + \delta[\mu] \circ h = id - p_\mu^* \circ i_\mu^*$$

hold. This implies Claim A.1.

Having established Claim A.1 the constructions of elements $\gamma_1, \gamma_2, \ldots$ with $\gamma_k \in \Gamma(\bigwedge^{(k+1)} \mathcal{E} \otimes \bigwedge^k \mathcal{E}^*)$ such that $\Omega_0 + p^!(\mu) + \gamma_1 + \gamma_2 + \cdots$ is a MC-element goes through as in the proof of Proposition 2 in Subsect. 3.3: One tries to extend $\Omega_0 + p^!(\mu)$ inductively and meets obstructions classes at each level. The first obstruction class vanishes if and only if S_μ is a coisotropic submanifold of E: $2R_0 := [\Omega_0 + p^!(\mu), \Omega_0 + p^!(\mu)]_{\iota_\nabla(\Pi)}$ gives a cohomology class in $H(BFV(E, \Pi), \delta_\mu)$, the evaluation of $2R_0$ at S_μ is 0 if and only if the vanishing ideal of S_μ is a Lie subalgebra under the Poisson bracket $[-, -]_\Pi$. This is equivalent to S_μ being coisotropic. When the class $[R_0]$

is zero, we can find γ_1 with $R_0 = -\delta_\mu(\gamma_1)$ which will be our first correction term. All higher obstruction classes vanish due to Claim A.1. Then setting $\beta_0 := p^!(\mu)$ and $\beta_m := \gamma_m - \Omega_m$ for $m > 1$ yields a MC-element

$$\beta := \sum_{k \geq 0} \beta_k$$

of the desired form.

Claim B. Given two elements α and β of $\mathcal{D}_{alg}(E, \Pi)$ with $T(\alpha) = T(\beta) = p^!(\mu)$ for some $\mu \in \Gamma(E)$, there is an element of $Inn^{\geq 2}(BFV(E, \Pi))$ mapping α to β.

Observe that inner derivations given by the adjoint action of $\widetilde{BFV}^0_{\geq 2}(E, \Pi)$ are nilpotent and therefore always integrate to an inner automorphism. Assume that β and α coincide up to order $k > 0$, i.e.

$$\beta - \alpha = 0 \text{ mod } BFV_{\geq k}(E, \Pi).$$

The MC-equation for β and α implies that

$$\delta[\mu](\beta_k) = F(\beta_0, \ldots, \beta_{(k-1)}) = F(\alpha_0, \ldots, \alpha_{(k-1)}) = \delta[\mu](\alpha_k).$$

Here F is a function that can be constructed from the MC-equation: the equation $1/2[\Omega + \beta, \Omega + \beta]_{BFV} = 0$ can be decomposed with respect to the ghost-momentum degree. For the ghost-momentum degree $k - 1$ one obtains $\delta[\mu](\beta_k)$ plus other terms depending on $(\beta_0, \ldots, \beta_{(k-1)})$ only. We denote the sum of these other terms by $-F$.

Consequently $\delta[\mu](\beta_k - \alpha_k) = 0$. By Claim A.1 and the assumption $k > 0$ there is an element $\varepsilon_k \in BFV^{(k+1,k+1)}(E, \Pi)$ such that $\beta_k - \alpha_k = \delta[\mu](\epsilon_k)$. Then

$$\begin{aligned} exp(-[\epsilon_k, -]_{BFV})(\alpha) &= \alpha - [\epsilon_k, \alpha]_{BFV} \text{ mod } BFV_{\geq(k+1)}(E, \Pi) \\ &= \alpha + [\alpha, \epsilon_k]_{BFV} \text{ mod } BFV_{\geq(k+1)}(E, \Pi) \\ &= \alpha + \delta[\mu](\epsilon_k) \text{ mod } BFV_{\geq(k+1)}(E, \Pi) \\ &= \beta \text{ mod } BFV_{\geq(k+1)}(E, \Pi) \end{aligned}$$

so $exp(-[\epsilon_k, -])(\alpha)$ and β coincide up to order $k + 1$.

Inductively one finds $\varepsilon_1, \epsilon_2, \ldots, \epsilon_N$ such that

$$exp(-[\epsilon_N, -]) \cdots exp(-[\epsilon_2, -])exp(-[\epsilon_1, -])(\alpha) = \beta.$$

Then the BCH-formula yields an $\varepsilon \in BFV^0_{\geq 2}(E, \Pi)$ such that the inner automorphism generated by its adjoint action on $BFV(E, \Pi)$ maps α to β. $\square$

5.3. An example. We consider an example that was first presented in [Z] and that was also investigated in [OP]. Zambon showed that the space of coisotropic deformations $\mathcal{C}(E, \Pi)$ "near" a fixed coisotropic submanifold S cannot carry the structure of a (Fréchet) manifold because there exist "tangent vectors" whose sum is not tangent to $\mathcal{C}(E, \Pi)$. Oh and Park showed that this can be understood with the help of the strong homotopy Lie algebroid $(\Gamma(\bigwedge E), \partial_s = \mu^1, \mu^2, \ldots)$ associated to S, see Definition 10 in Subsect. 4.1. The extension of elements in the first Lie algebroid cohomology to MC-elements meets several obstructions, and the first of them is given by a quadratic relation. Hence, the sum of solutions might fail to be a solution again which explains Zambon's observation.

Consider the vector bundle $E = \mathbb{R}^2 \times (S^1)^4 \to (S^1)^4$ with coordinates $(x^1, x^2, \theta^1, \theta^2, \theta^3, \theta^4)$ (θ denotes the angle-coordinate on S^1). We equip E with the symplectic form

$$\omega = d\theta^1 \wedge dx^1 + d\theta^2 \wedge dx^2 + d\theta^3 \wedge d\theta^4$$

and define $S := (S^1)^4$ which is a coisotropic submanifold of E.

The BFV-complex $BFV(E, \omega^{-1})$ is given by the smooth functions on the smooth graded manifold $E \times (\mathbb{R}^*[1])^2 \times (\mathbb{R}[-1])^2 \to E$. We introduce fibre coordinates (c_1, c_2) on $(\mathbb{R}^*[1])^2$ and (b^1, b^2) on $(\mathbb{R}[-1])^2$. Since the bundle E is flat we can just set

$$[-,-]_{BFV} =: [-,-]_G + [-,-]_\omega,$$

where $[-,-]_G$ denotes the graded Poisson bracket given by the pairing between $(\mathbb{R}^*[1])^2$ and $(\mathbb{R}[-1])^2$ and $[-,-]_\omega$ is the Poisson bracket associated to the symplectic form ω.

The element Ω_0 reads $c_1 x^1 + c_2 x^2$ and it is closed with respect to the graded Poisson bracket on the BFV-complex, so no further extension is needed and $\Omega = \Omega_0$. The BFV-differential D_{BFV} of the BFV-complex is given by

$$D = x^1 \frac{\partial}{\partial b^1} + x^2 \frac{\partial}{\partial b^2} + c^1 \frac{\partial}{\partial \theta^1} + c^2 \frac{\partial}{\partial \theta^2}.$$

It is straightforward to check that the cohomology with respect to D_{BFV} is given by periodic functions in the variables θ^3 and θ^4 tensored by the Grassmann-algebra generated by c^1 and c^2.

The MC-equation reads

$$[\Omega_0 + \beta, \Omega_0 + \beta]_{BFV} = 0,$$

and if we assume that β is a D_{BFV}-cocycle it reduces to

$$[\beta, \beta]_\omega = 0.$$

If we impose that β is a normalized MC-element (see Subsect. 5.2) it only depends on the variables $\theta^1, \theta^2, \theta^3$ and θ^4. In this case the MC-equation reduces further to

$$\{\beta, \beta\}_S = 0, \tag{9}$$

where

$$\{f, g\}_S := \frac{\partial f}{\partial \theta^4}\frac{\partial g}{\partial \theta^3} - \frac{\partial f}{\partial \theta^3}\frac{\partial g}{\partial \theta^4}.$$

Condition (9) was also found in [OP].

Consider an element $c_1 f^1 + c_2 f^2$, where f^1 and f^2 depend on the angle-variables only. When does this section define a coisotropic submanifold? In the proof of Proposition 6 in Subsect. 5.2 we showed that this is equivalent to

$$[\Omega_0 + c_1 f^1 + c_2 f^2, \Omega_0 + c_1 f^1 + c_2 f^2]_\omega \tag{10}$$

being exact with respect to $\delta[c_1 f^1 + c_2 f^2] := (x^1 + f^1)\frac{\partial}{\partial b^1} + (x^2 + f^2)\frac{\partial}{\partial b^2}$. Computing the bracket (10) yields

$$2c_1c_2\left(\frac{\partial f^1}{\partial \theta^2} - \frac{\partial f^2}{\partial \theta^1} + \frac{\partial f^1}{\partial \theta^4}\frac{\partial f^2}{\partial \theta^3} - \frac{\partial f^2}{\partial \theta^4}\frac{\partial f^1}{\partial \theta^3}\right).$$

We denote this expression by H. It only depends on the angle-variables. Exactness of H translates into the condition that there exists a pair of functions g_1 and g_2 that might depend on all variables on E such that

$$\delta[c_1 f^1 + c_2 f^2](b^1 g_1 + b^2 g_2) = (x^1 + f^1)g_1 + (x^2 + f^2)g_2 = H.$$

Since H is constant in x^1 and x^2 the left hand side $(x^1 + f^1)g_1 + (x^2 + f^2)g_2$ is too. Evaluating it at $x^1 = -f^1$ and $x^2 = -f^2$ implies that H must vanish identically. Hence a section of the bundle $\bigwedge(\mathbb{R}^2) \times E \to E$ given by $c_1 f^1 + c_2 f^2$ defines a coisotropic submanifold iff

$$\frac{\partial f^1}{\partial \theta^2} - \frac{\partial f^2}{\partial \theta^1} + \frac{\partial f^1}{\partial \theta^4}\frac{\partial f^2}{\partial \theta^3} - \frac{\partial f^2}{\partial \theta^4}\frac{\partial f^1}{\partial \theta^3} = 0.$$

Up to different sign conventions this condition coincides with the one given in [Z], where it was derived in an analytical context.

5.4. Formal deformations of coisotropic submanifolds. Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is coisotropic.

We introduce a formal parameter ε of degree 0 and consider the graded commutative algebra $BFV(E, \Pi)[[\varepsilon]]$. It inherits the structure of a differential graded Poisson algebra $(BFV(E, \Pi)[[\varepsilon]], D_{BFV}, [-, -]_{BFV})$ from $BFV(E, \Pi)$, see Definition 9 in Subsect. 3.3. We define the space of *formal MC-elements* by

$$\mathcal{D}_{for}(E, \Pi) := \{\beta \in \varepsilon BFV(E, \Pi)[[\varepsilon]] : [\Omega + \beta, \Omega + \beta]_{BFV} = 0\}.$$

Recall that Ω is a degree 1 element of $BFV(E, \Pi)$ such that $[\Omega, \Omega]_{BFV} = 0$ and if one decomposes Ω with respect to the ghost-momentum degree, i.e.

$$\Omega = \sum_{k \geq 0} \Omega_k$$

with $\Omega_k \in \Gamma(\bigwedge^{(k+1)} \mathcal{E} \otimes \bigwedge^k \mathcal{E}^*)$, Ω_0 is required to be the tautological section of $\mathcal{E} \to E$.

In Subsect. 5.2 we introduced $\widetilde{BFV}^0(E, \Pi)$ and its action by derivations on $BFV(E, \Pi)$. In the formal setting one considers $\varepsilon \widetilde{BFV}^0(E, \Pi)[[\varepsilon]]$ and its action on $BFV(E, \Pi)[[\varepsilon]]$. Since the action by such a derivation is pro-nilpotent, it always integrates to an automorphism of the graded Poisson algebra $(BFV(E, \Pi)[[\varepsilon]], [-, -]_{BFV})$. We denote the subgroup of these automorphisms by $Inn_{for}(BFV(E, \Pi))$. As explained in Subsect. 5.2 this group naturally acts on $\mathcal{D}_{for}(E, \Pi)$ by

$$\begin{aligned} Inn_{for}(BFV(E, \Pi)) \times \mathcal{D}_{for}(E, \Pi) &\to \mathcal{D}_{for}(E, \Pi) \\ (\Psi, \beta) &\mapsto \Psi(\Omega + \beta) - \Omega. \end{aligned}$$

Throughout Subsect. 5.2 we had to fix a normalization condition on the MC-elements of $(BFV(E, \Pi), D_{BFV}, [-, -]_{BFV})$ in order to make connection to the geometry of coisotropic submanifolds of (E, Π). We considered the truncation map $T : BFV^1(E, \Pi) \to \Gamma(\mathcal{E})$ and imposed that the image of a MC-element β under T has to lie in the image of the pull back map $\Gamma(E) \to \Gamma(\mathcal{E})$. In the formal setting no normalization condition is needed due to the following

Lemma 10. *For every $\beta \in \mathcal{D}_{for}(E,\Pi)$ there is a $\Psi \in Inn_{for}(BFV(E,\Pi))$ such that the image of $\Psi(\beta)$ under $T : BFV^1(S,\Pi)[[\varepsilon]] \to \Gamma(\mathcal{E})[[\varepsilon]]$ is given by a pull back from $\varepsilon\Gamma(E)[[\varepsilon]]$.*

Proof. The element $\beta \in \mathcal{D}_{for}(E,\Pi) \subset \varepsilon BFV^1(E,\Pi)[[\varepsilon]]$ decomposes uniquely into

$$\beta = \sum_{k\geq 0} \beta_k$$

with $\beta_k \in \varepsilon\Gamma(\bigwedge^{(k+1)}\mathcal{E} \otimes \bigwedge^k \mathcal{E}^*)[[\varepsilon]]$. In particular $\beta_0 \in \varepsilon\Gamma(\mathcal{E})[[\varepsilon]]$ which we further decompose as

$$\beta_0 = \sum_{l\geq 1} \beta_0(l)\varepsilon^l.$$

Consider the cocycle $[\beta_0(1)] \in H(BFV(E,\Pi),\delta)$. Using the homotopy h introduced in the proof of Proposition 2 in Subsect. 3.3 one finds a section $\tilde{\beta}_0(1) \in \varepsilon\Gamma(\mathcal{E})$ that is a pull back from a section of $\varepsilon\Gamma(E)$ such that $[\beta_0(1)] = [\tilde{\beta}_0(1)]$. Hence there is $\gamma(1) \in \varepsilon\Gamma(\mathcal{E}\otimes\mathcal{E}^*)$ satisfying $\beta_0(1) = \tilde{\beta}_0(1) + \delta(\gamma(1))$. The automorphism $exp([\gamma(1), -]_{BFV})$ maps the MC-element $\Omega + \beta$ to another one whose image under the truncation map modulo ε^2 is given by

$$\Omega_0 + \beta_0(1) - \delta(\gamma(1)) = \Omega_0 + \tilde{\beta}_0(1),$$

i.e. the new MC-element has the desired property modulo ε^2.

Let us assume that we established $\beta_0 = p^!(\mu)$ modulo ε^k for some $\mu \in \varepsilon\Gamma(E)[[\varepsilon]]$. Consider the δ-cocycle $\beta_0(k)$. As before there is $\gamma(k) \in \varepsilon^k\Gamma(\mathcal{E}\otimes\mathcal{E}^*)$ and a pull back section $\tilde{\beta}_0(k) \in \varepsilon^k\Gamma(\mathcal{E})$ such that

$$\beta_0(k) = \tilde{\beta}_0(k) + \delta(\gamma(k))$$

holds. We consider the inner automorphism $exp([\gamma(k), -]_{BFV})$ which maps the MC-element $\Omega + \beta$ to another one whose truncation modulo $\varepsilon^{(k+1)}$ is given by

$$\Omega_0 + \sum_{1\leq m\leq k} \beta_0(m) - \delta(\gamma(k)) = \Omega_0 + \sum_{1\leq m\leq (k-1)} \beta_0(m) + \tilde{\beta}_0(k).$$

Using induction with respect to the polynomial degree in ε, the fact that the formal variable ring is complete with respect to the ε-adic topology and the BCH-formula one finds an appropriate formal inner automorphism Ψ. □

In Subsect. 2.5 we stated that one possible characterization of coisotropic submanifolds uses vanishing ideals: a submanifold S of a Poisson manifold (E,Π) is coisotropic if and only if its vanishing ideal $\mathcal{I}_S := \{f \in \mathcal{C}^\infty(E) : f|_C = 0\}$ is a Lie subalgebra of the Poisson algebra of functions. A multiplicative ideal of a Poisson algebra that in addition is a Lie subalgebra is called a ***coisotrope***, see [W2].

Definition 13. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E,Π) is a Poisson manifold such that S is coisotropic.*

A **formal deformation** *of S is a coisotrope $\mathcal{I}$ of $(\mathcal{C}^\infty(E)[[\varepsilon]], [-,-]_\Pi)$ such that $\mathcal{I}$ mod $\varepsilon = \mathcal{I}_S$. We denote the set of formal deformations of S by $\mathcal{C}_{for}(E,\Pi)$.*

Lemma 11. *Let $E \to S$ be a finite rank vector bundle over a smooth finite dimensional manifold S. Assume (E, Π) is a Poisson manifold such that S is coisotropic.*
There is map Γ from $\mathcal{D}_{for}(E, \Pi)$ to $\mathcal{C}_{for}(E, \Pi)$ such that

- Γ *is constant along the $Inn^{\geq 1}_{for}(BFV(E, \Pi))$-orbits of $\mathcal{D}_{for}(E, \Pi)$,*
- $\Gamma(0)$ *is the $\mathbb{R}[[\varepsilon]]$-linear extension of the vanishing ideal $\mathcal{I}_S$ of S.*

Proof. Given $\beta \in \mathcal{D}_{for}(E, \Pi)$ we want to construct a coisotrope $\mathcal{I}(\beta)$ of $(\mathcal{C}^\infty(E)[[\varepsilon]], [-, -]_\Pi)$ in a way that is invariant under the action of the group $Inn^{\geq 1}_{for}(BFV(E, \Pi))$ on $\mathcal{D}_{for}(E, \Pi)$.

Consider the truncation $\beta_0 \in \varepsilon\Gamma(\mathcal{E})[[\varepsilon]]$ of β. We choose a trivializing atlas $(U_\alpha)_{\alpha \in A}$ for the vector bundle $E \to S$ which yields a trivializing atlas for the vector bundle $\mathcal{E} \to E$. On each chart U_α we pick a local frame $(c^\alpha_j)_{j=1,\dots,e}$ for the bundle $\mathcal{E}$ and obtain a unique decomposition

$$(\Omega_0 + \beta_0)|_{U_\alpha} = \sum_{j=1}^{e} h^j_\alpha c^\alpha_j$$

with $h^j_\alpha \in \mathcal{C}^\infty(U_\alpha \times \mathbb{R}^e)[[\varepsilon]]$ for $j = 1, \dots, e$. Let J_α be the multiplicative ideal of $\mathcal{C}^\infty(U_\alpha \times \mathbb{R}^e)[[\varepsilon]]$ generated by $(h^j_\alpha)_{j=1,\dots,e}$. It is straightforward to conclude from $\beta \in \mathcal{D}_{for}(E, \Pi)$ that J_α is a coisotrope of the Poisson algebra $(\mathcal{C}^\infty(U_\alpha \times \mathbb{R}^e)[[\varepsilon]], [-, -]_\Pi |_{U_\alpha \times \mathbb{R}^e})$.

Observe that the family of ideals $(J_\alpha)_{\alpha \in A}$ can be glued together, i.e. given $U_\alpha \cap U_\beta =: U_{\alpha\beta} \neq \emptyset$ then $f \in \mathcal{C}^\infty(U_{\alpha\beta} \times \mathbb{R}^e)$ lies in the restriction of J_α to $U_{\alpha\beta} \times \mathbb{R}^e$ if and only if it lies in the restriction of J_β to $U_{\alpha\beta} \times \mathbb{R}^e$. This stems from the fact that the transition matrices $U_{\alpha\beta} \times \mathbb{R}^e \to GL(\mathbb{R}^e)$ for the vector bundle $\mathcal{E}$ are invertible.

We define $\mathcal{I}(\beta)$ to be the set of elements of $\mathcal{C}^\infty(E)[[\varepsilon]]$ whose restriction to every coordinate domain $U_\alpha \times \mathbb{R}^e$ lies in J_α. An argument similar to the gluing statement above shows that $\mathcal{I}(\beta)$ is in fact independent of the choice of atlas and one easily checks that it is a coisotrope and that $\mathcal{I}(\beta) \bmod \varepsilon = \mathcal{I}_S$ holds.

Furthermore $\mathcal{I}(\beta)$ is not affected if we let a bundle automorphism act on the section β_0. Notice that the action of $\varepsilon BFV_{\geq 1}(E, \Pi)[[\varepsilon]]$ on $BFV(E, \Pi)[[\varepsilon]]$ induces an action on $BFV^{(1,0)}(E, \Pi)[[\varepsilon]] = \Gamma(\mathcal{E})[[\varepsilon]]$ which coincides with the action given by

$$BFV^{(1,1)}(E, \Pi) \to \Gamma(\mathcal{E} \otimes \mathcal{E}^*) \cong \Gamma(End(\mathcal{E})) \xrightarrow{exp} \Gamma(GL(\mathcal{E})).$$

From this the $Inn^{\geq 1}_{for}(BFV(E, \Pi))$-equivariance of $\beta \mapsto \mathcal{I}(\beta)$ follows. □

Appendix A. Details on the Homotopy Transfer

This Appendix provides background information on the material presented in Subsect. 2.3. The aim is to prove Theorem 2 which is a central technical tool in Sects. 3 and 4. We first relate the homotopy transfer to integration over an isotropic subspace in the BV-Formalism. Then we check that the formulae given in 2.3 actually work. All the results are well-known to the experts and we do not claim any originality related to this treatment.

A.1. Connection to the BV-formalism. We present a heuristic derivation of the formulae for the homotopy transfer as presented in Subsect. 2.3. It makes use of the BV-Formalism which was introduced by Batalin and Vilkovisky. In the case of A_∞-algebras a similar treatment can be found in [Ka].

In the finite dimensional setting the BV-Formalism was made rigorous by Schwarz, see [Sw]. Although not justified at a mathematical level of rigor in the infinite-dimensional setting in general, the BV-Formalism serves as a way to obtain formulae for the homotopy transfer which can be checked to work using purely algebraic manipulations a posteriori. We remark that there are certain (infinite-dimensional) situations where a mathematical treatment can be provided—see [Co] for instance.

Let V be a graded vector space. In Subsect. 2.1 the one-to-one correspondence between L_∞-algebra structures on V and codifferentials of $S(V[1])$ was explained. If one assumes that V is finite dimensional, the space of coderivations of the coalgebra $S(V[1])$ is in bijection to the space of derivations of the algebra $S(V^*[-1]) =: \mathcal{C}^\infty(V[1])$, i.e. vector fields on $V[1]$. Under this bijection codifferentials correspond to so called *cohomological vector fields*, i.e. derivations of degree 1 that square to zero. Hence there is a one-to-one correspondence between L_∞-algebra structures on V and homological vector fields on $V[1]$. Moreover flat L_∞-algebras are encoded in homological vector fields that vanish at $0 \in V[1]$.

The space of multivector fields on $V[1]$ can be described as the space of functions on the smooth graded manifold $T^*[1](V[1]) = V[1] \oplus V^*[0]$. Being a shifted cotangent bundle, this smooth graded manifold carries a graded symplectic structure. Equivalently the graded commutative algebra $\mathcal{C}^\infty(T^*[1](V[1]))$ carries the structure of a graded Poisson bracket $[-,-]_{BV}$ of degree -1 called the *BV-bracket*. The space of vector fields forms a Poisson subalgebra and a vector field X on $V[1]$ is cohomological if and only if $[X, X]_{BV} = 0$. This equation is called the *classical master equation*.

There is a bijection between the space of homomorphisms $Hom(V[1], V)$ of $V[1]$ of degree -1 and the graded vector space $V^*[-1] \otimes V[0]$. Using a basis (γ_i) of $V[0]$ and the dual basis (γ^i) of $V^*[-1]$ the identity $id \in End(V)$ yields an element $\gamma^i \otimes \gamma_i$. One defines the following operator of degree -1 on $\mathcal{C}^\infty(V[1] \oplus V^*[0])$:

$$\Delta := \frac{\partial^2}{\partial \gamma^i \partial \gamma_i}$$

which is called the *BV-operator*. It is straightforward to check that $\Delta \circ \Delta = 0$. However Δ is not a derivation with respect to the graded commutative associative product of $\mathcal{C}^\infty(V[1] \oplus V^*[0])$. The deviation to being a derivation is measured by the BV-bracket $[-,-]_{BV}$, i.e.

$$\Delta(X \cdot Y) - \Delta(X) \cdot Y - (-1)^{|X|} X \cdot \Delta(Y) = (-1)^{|X|}[X, Y]_{BV}$$

for homogeneous X and arbitrary Y in $\mathcal{C}^\infty(V[1] \oplus V^*[0])$. The quadruple $(\mathcal{C}^\infty(V[1] \oplus V^*[0]), \cdot, \Delta, [-,-]_{BV})$ is an example of a *BV-algebra*. Given such an algebra one can write down the *quantum master equation*:

$$\Delta(X) + \frac{1}{2}[X, X]_{BV} = 0.$$

The importance of this equation is due to the identity

$$\Delta(e^X) = (\Delta(X) + \frac{1}{2}[X, X]_{BV})e^X.$$

Hence e^X is Δ-closed if and only if X satisfies the quantum master equation.

Let X be a cohomological vector field on a graded vector space $V[1]$ which vanishes at 0, i.e. $V[1]$ is equipped with the structure of a flat $L_\infty[1]$-algebra. Denote the differential of this $L_\infty[1]$-structure by δ and the corresponding cohomology by $H[1]$. Suppose that there are chain maps $i : H[1] \hookrightarrow V[1]$ (injective) and $p : V[1] \to H[1]$ (surjective) such that $p \circ i = id_{H[1]}$. Hence $V[1]$ splits as a graded vector space into $A[1] \oplus H[1]$. We assume existence of a homotopy $h : V[1] \to V$ such that

$$\delta \circ h + h \circ \delta = id_{V[1]} - i \circ p.$$

The kernel of this map is a graded vector subspace of $V[1]$. We consider its intersection with $A[1]$ which we denote by $K[1]$. The conormal bundle $L[1]$ of $K[1]$ as a graded vector subspace of $A[1]$ is a Lagrangian vector subspace of $T^*[1](A[1])$ and an isotropic subspace of $T^*[1](V[1])$.

Given a Lagrangian vector subspace $L[1]$ of $T^*[1](A[1])$ there is a well-defined notion of integration

$$\int_{L[1]} : \mathcal{C}^\infty(T^*[1](A[1]) \to \mathbb{R}$$

under suitable convergence assumptions, see [Sw]. The connection between the quantum master equation and the integration theory is

Theorem 7. • *Assume $S \in \mathcal{C}^\infty(T^*[1](A[1]))$ is Δ-closed and let $L[1]$ and $L'[1]$ be two cobordant Lagrangian submanifolds of $T^*[1](A[1])$. Then $\int_{L[1]} S = \int_{L'[1]} S$.*

• *Assume $S \in \mathcal{C}^\infty(T^*[1](A[1]))$ is Δ-exact and let $L[1]$ be any Lagrangian submanifold of $T^*[1](A[1])$. Then $\int_{L[1]} S = 0$.*

The proof in the finite dimensional setting can be found in [Sw].

Using the splitting $V[1] = A[1] \oplus V[1]$ and the induced splitting of $T^*[1](V[1])$ one can extend $\int_{L[1]}$ to a map

$$\int_{L[1]} : \mathcal{C}^\infty(T^*[1](V[1])) \to \mathcal{C}^\infty(T^*[1](H[1])).$$

Furthermore the BV-operator Δ also decomposes into $\Delta_A + \Delta_H$. Due to Theorem 7, $\int_{L[1]}$ is a chain map between the complexes $(\mathcal{C}^\infty(T^*[1](V[1])), \Delta)$ and $(\mathcal{C}^\infty(T^*[1](H[1])), \Delta_{H[1]})$.

One can apply the BV-Formalism as follows: interpret a vector field Z on $V[1]$ as a function on $T^*[1](V[1])$ and assume that it satisfies the quantum master equation. Hence e^Z is Δ-closed. Apply the map $\int_{L[1]}$ to obtain a function Y' on $T^*[1](H[1])$ that satisfies the quantum master equation with respect to Δ_H. If one assumes that there is a function Z' such that $e^{Z'} = Y'$ it follows that Z' is a vector field that satisfies the quantum master equation. This procedure has a physical interpretation in terms of integrating out ultraviolet degrees of freedom. Moreover there is a purely algebraic interpretation of the integration map $\int_{L[1]}$ in terms of certain graphs, known as Feynman diagrams.

It can be physically justified that in the "classical limit" the whole procedure reduces to the following: start with a cohomological vector field X on $V[1]$, translate it to a function on $T^*[1](V[1])$. Using the tree-level part of the Feynman diagrams to "integrate" over the isotropic subspace $L[1]$ one obtains a cohomological vector field on $H[1]$. If one reinterprets this in terms of $L_\infty[1]$-algebra structures one recovers the procedure for homological transfer along contractions as presented in Subsect. 2.3.

Going beyond the tree-level in this integration procedure yields richer structures, see [Co and Mn] for instance.

A.2. Transfer of differential complexes.

Lemma 12. *Let (X, d, h, i, pr) be a graded vector space equipped with contraction data and a finite compatible filtration, i.e. a collection of graded subvector spaces*

$$X = \mathcal{F}_0 X \supseteq \mathcal{F}_1 X \supseteq \cdots \supseteq \mathcal{F}_n X \supseteq \mathcal{F}_{(n+1)} X \supseteq \cdots$$

such that $\mathcal{F}_N X = \{0\}$ for N large enough, satisfying

- *$d(\mathcal{F}_k X) \subset \mathcal{F}_k X$ for all $k \geq 0$ and*
- *$h(\mathcal{F}_k X) \subset \mathcal{F}_k X$ for all $k \geq 0$.*

Furthermore suppose X is equipped with the structure of a differential complex (X, D) such that

- *$(D - d)(\mathcal{F}_k X) \subset \mathcal{F}_{(k+1)} X$.*

Then the cohomology H of (X, d) is naturally equipped with the structure of a differential complex and there is a well-defined chain map $\tilde{i} : H \to X$.

Proof. Set $D_R := D - d$; it follows from $D^2 = (d + D_R)^2 = 0$ and $d^2 = 0$ that

$$D_R \circ d + d \circ D_R + D_R^2 = 0$$

holds. In this special case the formulae for the induced structure given in Subsect. 2.3 reduce to

$$\mathcal{D} := p \circ \tilde{\mathcal{D}} \circ i, \text{ where}$$
$$\tilde{\mathcal{D}} := D_R \left(\sum_{k \geq 0} (-h D_R)^k \right).$$

Claim 1. $\mathcal{D} \circ \mathcal{D} = 0$.

We compute

$$\begin{aligned}
-d(\tilde{\mathcal{D}}) &= D_R D_R \left(\sum_{m \geq 0} (-h D_R)^m \right) + D_R d \left(\sum_{m \geq 0} (-h D_R)^m \right) \\
&= D_R D_R \left(\sum_{m \geq 0} (-h D_R)^m \right) + D_R i p \left(D_R \circ \sum_{m \geq 0} (-h D_R)^m \right) \\
&\quad - D_R D_R \left(\sum_{m \geq 0} (-h D_R)^m \right) + D_R h d \left(D_R \circ \sum_{m \geq 0} (-h D_R)^m \right) \\
&= \tilde{\mathcal{D}} i p \tilde{\mathcal{D}} + \tilde{\mathcal{D}} d,
\end{aligned}$$

and consequently

$$\mathcal{D}^2 = p \tilde{\mathcal{D}} i \circ p \tilde{\mathcal{D}} i = 0.$$

The formulae for the $L_\infty[1]$-morphism given in Subsect. 2.3 reduce to

$$\tilde{i} := \left(\sum_{k\geq 0}(-hD_R)^k\right) i.$$

Claim 2. $\tilde{i}$ is a chain map from $(H, \mathcal{D})$ to (X, D). First we rewrite $\tilde{i}$ as

$$\tilde{i} = (id - h\tilde{\mathcal{D}})i$$

and compute

$$\begin{aligned} D\circ\tilde{i} &= (d + D_R)(id - h\tilde{\mathcal{D}})i = d(-h\tilde{\mathcal{D}})i + \tilde{\mathcal{D}}i \\ &= ip\tilde{\mathcal{D}}i + hd(\tilde{\mathcal{D}})i = (id - h\tilde{\mathcal{D}})i \circ p\tilde{\mathcal{D}}i = \tilde{i}\circ\mathcal{D}. \end{aligned}$$

□

A.3. Transfer of differential graded Lie algebras. We prove Theorem 2, Subsect. 2.3: We are given contraction data (X, d, h, i, p) and the structure of a differential graded Lie algebra $(X, D, [-,-])$. In Subsect. A.2 we set $D_R := D - d$ and defined $\tilde{\mathcal{D}}$ and $\mathcal{D}$ respectively. We use the décalage-isomorphism to translate the graded Lie bracket into a graded symmetric operation which we denote by $\{-,-\}$ from now on.

The description of the induced structure maps can be rephrased as follows: consider an oriented trivalent tree T with n leaves whose edges are decorated by non-negative integers as introduced in Subsect. 2.3. One can associate a map $\Phi(T) := \tilde{m}_T : (X[1])^{\otimes n} \to X[1]$ to T by placing $\{-,-\}$ at its trivalent vertices, copies of D_R at all its edges and $-h$ between two consecutive such operations. Let

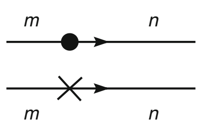
$$\tilde{\nu}^k := \sum_{\sigma\in\Sigma_k}\sum_{[T]\in[\mathbb{T}](k)} \frac{1}{|Aut(T)|}\tilde{m}_T$$

and observe that ν^k from Subsect. 2.3 coincides with $p\circ\tilde{\nu}^k\circ i^{\otimes k}$.

In Subsect. 2.1 we introduced the family of Jacobiators associated to a family of maps. By definition a family of maps constitutes an $L_\infty[1]$-algebra structure if the associated Jacobiators vanish. Denote the family of Jacobiators associated to $(\nu^k : S^k(H[1]) \to H[2])$ by (J^n). We can write $J^n = p\circ\tilde{J}^n\circ i^{\otimes n}$ with

$$\begin{aligned} &\tilde{J}^n(x_1\cdots x_n) \\ &:= \sum_{r+s=n}\ \sum_{\sigma\in(r,s)-\text{shuffles}} \mathrm{sign}(\sigma)\,\tilde{\nu}^{s+1}(ip\tilde{\nu}^r(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)}). \end{aligned}$$

Claim A. $-d\left(\sum_{\sigma\in\Sigma_k}\sum_{[T]\in[[\mathbb{T}]](n)}\frac{1}{|Aut(T)|}\sigma^*\tilde{m}_T\right)i^{\otimes n} = \tilde{J}^n i^{\otimes n}$. To prove this claim we introduce an extended graphical calculus: we allow to add one special edge in every tree which is marked either by a "·" of "×" and require that the special edge is decorated by two non-negative integers:



We call oriented decorated trees with a special edge of the first kind *pointed* and with a special edge of the second kind *truncated*. Denote the space of pointed oriented decorated trees by $\mathbb{T}^{\circ}$ and the space of truncated oriented decorated trees by $\mathbb{T}^{\times}$. We extend Φ to trees with marked special edges: instead of composing two consecutive operations of degree 1 by $\circ - h \circ$ we use $\circ i p \circ$ at the pointed edge and ordinary composition at the edge with a cross. Moreover one has to add the sign given by (-1) to the powers of the sum of all inputs left to the truncated or pointed edge.

One can easily check that

$$\sum_{\sigma \in \Sigma_k} \sum_{[T] \in [[\mathbb{T}]](n)} \frac{1}{|Aut(T)|} \sigma^* \Phi(P(T)) = \tilde{J}^n$$

holds where $P(T)$ is the sum of all ways to change an ordinary edge of T into a pointed one. Consequently Claim A follows from

Claim A.1.

$$-d \left(\sum_{\sigma \in \Sigma_n} \sum_{[T] \in [[\mathbb{T}]](n)} \frac{1}{|Aut(T)|} \sigma^* \tilde{m}_T \right) i^{\otimes n} = \left(\sum_{\sigma \in \Sigma_k} \sum_{[T] \in [[\mathbb{T}]](n)} \frac{1}{|Aut(T)|} \sigma^* \Phi(P(T)) \right) i^{\otimes n}.$$

We prove Claim A.1. by induction over the number of leaves n. For $n = 1$ the claim is simply the equation

$$-d\tilde{\mathcal{D}}i = \tilde{\mathcal{D}}ip\tilde{\mathcal{D}}i,$$

which was established in Subsect. A.2. The inductive step uses the identities

$$-d\Phi(\xrightarrow{\;n\;}) = \sum_{r+s=n+1} \Phi(\overset{r}{\times}\!\xrightarrow{\;s\;}) - \sum_{r+s=n} \Phi(\overset{r}{\times}\!\xrightarrow{\;s\;}) + \sum_{r+s=n} \Phi(\overset{r}{\bullet}\!\xrightarrow{\;s\;}) + \Phi(\xrightarrow{\;n\;})d$$

and

$$\begin{aligned} -d\{X, Y\} = {} & \{dX, Y\} + (-1)^{|X|}\{X, dY\} + \\ & + \{D_R X, Y\} + (-1)^{|X|}\{X, D_R Y\} + D_R\{X, Y\}. \end{aligned}$$

Computing the left hand side of the equation in Claim A.1, successively leads to the right-hand side plus

$$\sum_{\sigma \in \Sigma_n} \sum_{[T] \in [\mathbb{T}](n)} \frac{1}{|Aut(T)|} \sigma^* \Phi(X(T)),$$

where $X(T)$ is the sum of all ways to change an ordinary interior edge of T into a truncated one which is decorated by (0, 0). The evaluation of this sum at $x_1 \otimes \cdots \otimes x_n$ contains terms of the form

$$\begin{aligned} & \sum_{\sigma \in \Sigma_n} \operatorname{sign}(\sigma) \sum_{r+s+t=n} 1/2 \sum_{[U] \in [\mathbb{T}](r), [V] \in [\mathbb{T}](s), [W] \in [\mathbb{T}](t)} (\{\{-h \circ \Phi(U)(x_{\sigma(1)} \cdots x_{\sigma(r)}), \\ & \quad -h \circ \Phi(V)(x_{\sigma(r+1)} \cdots x_{\sigma(r+s)})\}, -h \circ \Phi(W)(x_{\sigma(r+s+1)} \cdots x_{\sigma(n)})\}). \end{aligned}$$

Since the expression in the last two lines is of the form $\{\{a, b\}, c\}$ and the sum runs over all permutations with appropriate signs it vanishes due to the graded Jacobi identity.

Hence $J^n = p\tilde{J}^n i^{\otimes n} = p(d(\dots)) = 0$ and consequently the induced structure maps $(\nu^k : S^k(H[1]) \to H[2])$ define an $L_\infty[1]$-algebra structure on $H[1]$.

It remains to show that the maps $\lambda^n : S^n(H[1]) \to X[1]$ defined in Subsect. 2.3 establish an $L_\infty[1]$-morphism between $(H[1], \nu^2, \nu^2, \dots)$ and $(X[1], D, \{-,-\})$. We give explicit formulae for the identities that must be checked in order to prove that we obtain an L_∞-morphism:

$$
\begin{aligned}
&-D(h \circ \tilde{\nu}^n(x_1 \otimes \cdots \otimes x_n)) \\
&+ 1/2 \sum_{r+s=n} \sum_{\sigma \in (r,s)-\text{shuffles}} \operatorname{sign}(\sigma)\{h \circ \tilde{\nu}^r(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}), \\
&\quad \times h \circ \tilde{\nu}^s(x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(n)})\} \\
&+ \sum_{p+q=n} \sum_{\tau \in (q,p)-\text{shuffles}} \operatorname{sign}(\tau) h \circ \tilde{\nu}^{p+1}(ip \circ \tilde{\nu}^q(x_{\tau(1)} \otimes \cdots \otimes x_{\tau(q)}) \\
&\quad \otimes x_{\tau(q+1)} \otimes \cdots \otimes x_{\tau(n)}) \\
&-ip\tilde{\nu}^n(x_1 \otimes \cdots \otimes x_n)
\end{aligned}
$$

has to vanish identically for all $n \geq 2$ (the case $n = 1$ was dealt with in Subsect. A.2).

It is straightforward to check that the expression

- in the second and third line is equal to $B := \tilde{\nu}^n + D_R h \tilde{\nu}^n$,
- in the fourth line is equal to $C := h\left(\sum_{\sigma \in \Sigma_n} \sum_{[T] \in [[\mathbb{T}]](n)} \frac{1}{|Aut(T)|} \sigma^* \Phi(P(T))\right)$,
- in the first line is equal to $-\tilde{\nu}^n + ip\tilde{\nu}^n + hd\tilde{\nu}^n - D_R h \tilde{\nu}^n$.

The identity $-d\tilde{\nu}^n i^{\otimes i} = \left(\sum_{\sigma \in \Sigma_n} \sum_{[T] \in [[\mathbb{T}]](n)} \frac{1}{|Aut(T)|} \sigma^* \Phi(P(T))\right) i^{\otimes n}$ implies that everything cancels.

Acknowledgements. The author acknowledges partial support by the joint graduate school of mathematics of the ETH and the University of Zürich, by SNF-grant Nr.20-113439, by the European Union through the FP6 Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652), and by the European Science Foundation through the MISGAM program. Moreover he thanks the ESI for Mathematical Physics for support during the author's visit in July and August 2007.

I thank A. Cattaneo for many encouraging and inspiring discussions and his general support. I also thank D. Fiorenza, D. Indelicato, B. Keller, P. Mnëv, T. Preu, C. Rossi, S. Shadrin, J. Stasheff and M. Zambon for clarifying discussions and helpful remarks on a draft of this paper. Moreover I thank M. Bordemann and H.-C. Herbig for pointing me to their globalization of the BFV-complex. The referee contributed a lot to the form of this work with his/her insightful suggestions.

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Communicated by N. A. Nekrasov